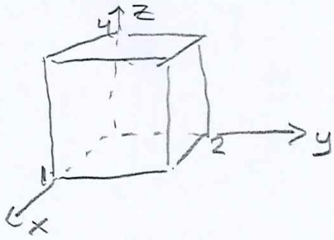
Bilder och tips 6.16-19 (lösningar efter)

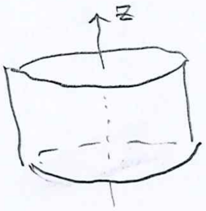
6.16 $D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \\ 0 \le z \le 4 \end{cases}$



$I = \iiint_D (\underbrace{\sqrt[3]{x}}_{\le 1} + \underbrace{y}_{\le 2} + \underbrace{\sqrt{z}}_{\le 2})\,dxdydz$

$\Rightarrow I \le 5 \iiint_D dxdydz = \ldots$

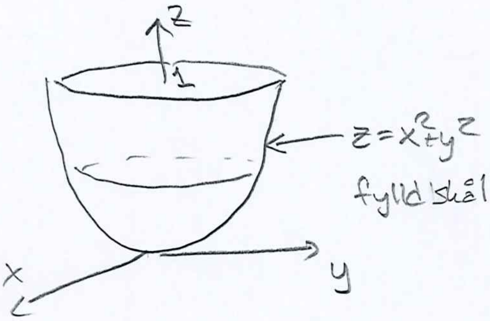
6.17 D cylinder, höjd 2, radie 3, z-axeln = symmetriaxel



$I = \iiint_D (x^2+y^2)\, z\, dxdydz$ $\qquad D: \begin{cases} x^2+y^2 \le 3^2 \\ a \le z \le a+2 \end{cases}$ ← OBS

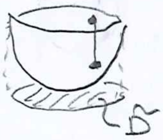
OBS: Det står <u>inte</u> i uppgiften att $0 \le z \le 2$, bara höjd 2.
Man får räkna med $a \le z \le a+2$ (a godtyckligt)
Polära koordinater är lämpligt i xy-led

6.18



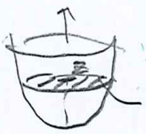
$I = \iiint_D f(x,y,z)\, dxdydz$

Om man vill integrera z <u>först</u>,
får man $x^2+y^2 \le z \le 1$



Sedan gäller $(x,y) \in \tilde{D}$ = proj. av D i xy-planet, dvs $\tilde{D}$ = enhetscirkelskiva

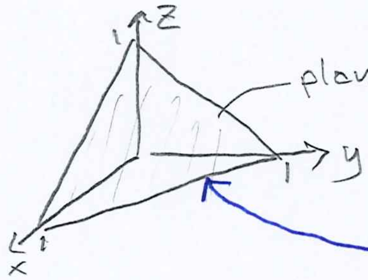
Om man vill integrera z sist får man $0 \le z \le 1$. Då ska först x, y integreras över ett snitt D_z av D på höjden z



$D_z : x^2+y^2 \le z$

Polära koordinater rekommenderas i xy-led

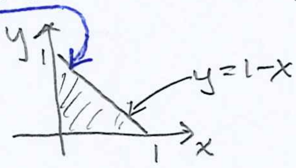
6.19 D är en tetraeder (fylld)



plan $x+y+z=1$ (går genom (1,0,0), (0,1,0) och (0,0,1))

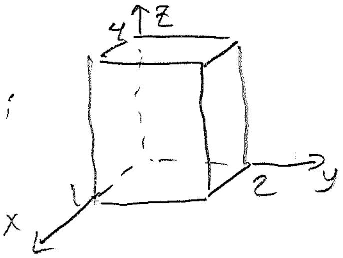
Om z integreras först: $0 \le z \le 1-x-y$

Kvar blir triangel i xy-planet



Lösningar

6.16

D: 

$$I=\iiint_D (\underbrace{\sqrt[3]{x}}_{\le 1}+\underbrace{y}_{\le 2}+\underbrace{\sqrt{z}}_{\le 2})\,dxdydz \le \iiint_D 5\,dxdydz =$$ inom D

$$=5\iiint_D 1\cdot dxdydz = 5\cdot \text{volym}(D) = 5\cdot 1\cdot 2\cdot 4 = 40$$

$\Rightarrow I \le 40$ nödvändigt

Exakt

$$I=\int_0^4\int_0^2\int_0^1 (x^{1/3}+y+z^{1/2})\,dxdydz = \int_0^4\int_0^2 \left[\tfrac34 x^{4/3}+xy+xz^{1/2}\right]_{x=0}^1 dydz = \int_0^4\int_0^2 \left(\tfrac34+y+z^{1/2}\right)dydz =$$

$$=\int_0^4 \left[\tfrac34 y+\tfrac12 y^2+yz^{1/2}\right]_{y=0}^2 dz = \int_0^4 \left(\underbrace{\tfrac32+2}_{7/2}+2z^{1/2}\right)dz = \left[\tfrac72 z+\tfrac43 z^{3/2}\right]_0^4 = 14+\tfrac43\underbrace{4\sqrt4}_{8} = 14+\tfrac{32}{3} = \underline{\tfrac{74}{3}}$$

6.17

$$I=\int_a^{a+2}\left(\iint_{\tilde D}(x^2+y^2)\,dxdy\right)z\,dz = \Big/\ \tilde D: x^2+y^2\le 3^2,\ \text{polära } \rho\le 3,\ 0\le\varphi<2\pi,\ \text{determinant}=\rho\ \Big/ = \int_a^{a+2}\left(\int_0^{2\pi}\int_0^3 \rho^2\cdot\rho\,d\rho d\varphi\right)z\,dz =$$

$$=\int_a^{a+2}\left(\int_0^{2\pi}\underbrace{\left[\tfrac{\rho^4}{4}\right]_0^3}_{=81/4}d\varphi\right)z\,dz = \frac{81}{4}\int_a^{a+2}\underbrace{[\varphi]_0^{2\pi}}_{=2\pi}z\,dz = \frac{81}{4}\cdot 2\pi\left[\frac{z^2}{2}\right]_a^{a+2} = \frac{81\pi}{4}\Big(\underbrace{(a+2)^2-a^2}_{4a+4}\Big) = 81(a+1)\pi$$

6.18

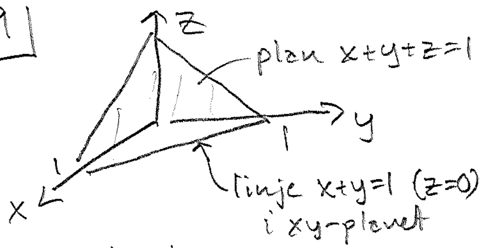
(Se tipsen) z först $\Rightarrow I=\iint_{\tilde D}\left(\int_{x^2+y^2}^1 f(x,y,z)\,dz\right)dxdy$, $\tilde D: x^2+y^2\le 1 \Rightarrow \begin{cases}0\le\rho\le 1\\ 0\le\varphi<2\pi\\ \rho^2\le z\le 1\end{cases}$ (1)

z sist $\Rightarrow I=\int_0^1\left(\iint_{D_z} f(x,y,z)\,dxdy\right)dz$, $D_z: x^2+y^2\le z \Rightarrow \begin{cases}0\le z\le 1\\ 0\le\varphi<2\pi\\ 0\le\rho\le\sqrt z\end{cases}$ (2) [polära för x,y]

(1) t ex ger $I=\int_0^{2\pi}\int_0^1\Big(\int_{\rho^2}^1 \underbrace{z\cdot\rho}_{f=z\sqrt{x^2+y^2}\text{ insatt}}\,dz\Big)\rho\,d\rho d\varphi$ determinant $=\int_0^{2\pi}\int_0^1\left[\frac{z^2}{2}\right]_{z=\rho^2}^1\rho^2\,d\rho d\varphi = \int_0^{2\pi}\int_0^1\left(\frac12-\frac{\rho^4}{2}\right)\rho^2\,d\rho d\varphi =$

$$=\frac12\int_0^{2\pi}\int_0^1(\rho^2-\rho^6)\,d\rho d\varphi = \frac12\int_0^{2\pi}\left[\tfrac13\rho^3-\tfrac17\rho^7\right]_0^1 d\varphi = \frac12\int_0^{2\pi}\underbrace{\left(\tfrac13-\tfrac17\right)}_{4/21}d\varphi = \frac12\cdot\frac{4}{21}\underbrace{[\varphi]_0^{2\pi}}_{2\pi} = \underline{\frac{4\pi}{21}}$$

6.19



$$D: \begin{cases}0\le z\le 1-x-y\\ 0\le y\le 1-x\\ 0\le x\le 1\end{cases} \Rightarrow$$

$$I=\int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}\right)dy\right)dx = \int_0^1\left(\int_0^{1-x}\left[\frac{-1/2}{(1+x+y+z)^2}\right]_{z=0}^{z=1-x-y}dy\right)dx = \int_0^1\left(\int_0^{1-x}\left(\frac{-1/2}{2^2}+\frac{1/2}{(1+x+y)^2}\right)dy\right)dx$$

$$=\int_0^1\left[-\tfrac18 y-\frac{1/2}{1+x+y}\right]_{y=0}^{y=1-x}dx = \int_0^1\left(-\tfrac18(1-x)-\frac{1/2}{2}+0+\frac{1/2}{1+x}\right)dx = \int_0^1\left(\frac x8-\frac38+\frac{1/2}{1+x}\right)dx =$$

$$=\left[\frac{x^2}{16}-\frac{3x}{8}+\frac12\ln|1+x|\right]_0^1 = \frac1{16}-\frac38+\frac12\ln 2-0+0-\frac12\underbrace{\ln 1}_{=0} = \underline{\frac12\ln 2-\frac{5}{16}}$$