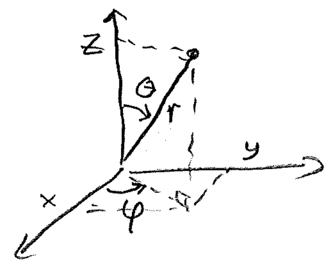
Tips & lösn. 6.26, 6.30

TIPS

6.26 Kom ihåg rymdpolära (=sfäriska) koordinater



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{matrix} 0 \le r < \infty \\ 0 \le \theta \le \pi \\ 0 \le \varphi < 2\pi \end{matrix}$$

determinant $\frac{d(x,y,z)}{d(r,\theta,\varphi)} = r^2\sin\theta \ (\geq 0)$
(behöver man inte härleda)

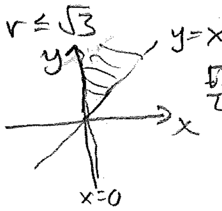
a) $x^2+y^2 = r^2\sin^2\theta$

$$\int(\cos^2\theta - \underbrace{\sin^2\theta}_{1-\cos^2\theta})\sin\theta\,d\theta = \int(\underbrace{2\cos^2\theta\sin\theta}_{\text{kan sätta } \cos\theta = t} - \sin\theta)\,d\theta = -\frac{2}{3}\cos^3\theta + \cos\theta \ (+C)$$

b) $x \geq 0$ är $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ (man får använda negativa φ för att skriva $x \geq 0$ som bara ett intervall i φ) $\left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$\int r^3 e^{r^2}\,dr$, partiell integration 3 gånger! Eller $t = r^2$ ($\Rightarrow$ bara en partiell integration)

c) $\begin{cases} x^2+y^2+z^2 \le 3 \Rightarrow 0 \le r \le \sqrt{3} \\ 0 \le x \le y \end{cases} \longrightarrow$



$\underbrace{\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}}$

Fullt θ-intervall: $0 \le \theta \le \pi$

6.30 $\text{Volym}(D) = \iiint_D 1 \cdot dx\,dy\,dz$ (def. av volym). Volym av klot är $\frac{4\pi R^3}{3}$ (R = radie)

a) Sätt $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c} \Rightarrow \ldots$, $\frac{d(x,y,z)}{d(u,v,w)} = ?$

b) Kvadratkomplettera och gör sedan linjärt variabelbyte ($\Rightarrow$ konstant determinant). Lämpligt börja med z-termer vid kvadratkompl.

$$3x^2+2y^2+\underbrace{z^2+2xz-2yz} = 3x^2+2y^2+\underbrace{(z+x-y)^2-x^2-y^2+2xy} =$$

$$= (\underbrace{z+x-y}_{u})^2 + 2x^2 + \underbrace{y^2+2xy}_{(y+x)^2-x^2} = \ldots \qquad \begin{cases} u = \ldots \\ v = \ldots \\ w = \ldots \end{cases}$$

[volymsformeln $\frac{4\pi R^3}{3}$ för klot härleds med rymdpolära koordinater i anteckningarna, föreläsning 14]

LÖSNINGAR

6.26 a) Enhetsklotet i rymdpolära koord: $\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \pi \\ 0 \le \varphi < 2\pi \end{cases}$, $\dfrac{d(x,y,z)}{d(r,\theta,\varphi)} = r^2 \sin\theta$

$$\iiint_D (z^2 - x^2 - y^2)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} (r^2\cos^2\theta - r^2 \underbrace{\sin^2\theta}_{1-\cos^2\theta})\, r^2 \sin\theta \, dr\, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 (2\cos^2\theta - 1)\sin\theta \, dr\, d\theta\, d\varphi$$

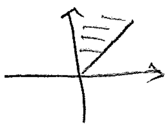
$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{\left[\frac{r^5}{5}\right]_0^1}_{=1/5} (2\cos^2\theta \sin\theta - \sin\theta)\, d\theta\, d\varphi = \frac{1}{5}\int_0^{2\pi} \underbrace{\left[-\frac{2}{3}\cos^3\theta + \cos\theta\right]_0^{\pi}}_{=\frac{2}{3}-1+\frac{2}{3}-1=-\frac{2}{3}} d\varphi = -\frac{2}{3}\cdot\frac{1}{5}\underbrace{\left[\varphi\right]_0^{2\pi}}_{=2\pi} = \underline{-\frac{4\pi}{15}}$$

6.26 b) $D: \begin{cases} x^2+y^2+z^2 \le 1 \\ x \ge 0 \end{cases}$, rymdpolära $\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \pi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases}$

$$I = \iiint_D x e^{x^2+y^2+z^2}\,dx\,dy\,dz = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{1} r\sin\theta\cos\varphi\, e^{r^2} \underbrace{r^2\sin\theta}_{\text{determinant}}\, dr\, d\theta\, d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi} \underbrace{\int_0^1 r^3 e^{r^2} dr}\, \sin^2\theta\, d\theta \cos\varphi\, d\varphi = \left[\begin{array}{l} t = r^2 \\ dt = 2r\,dr \\ r=0 \Rightarrow t=0,\ r=1 \Rightarrow t=1 \end{array}\right. \int_0^1 r^3 e^{r^2} dr = \frac{1}{2}\int_0^1 t e^t dt = \frac{1}{2}\left(\underbrace{\left[te^t\right]_0^1}_{e^1-0} - \underbrace{\int_0^1 1 e^t dt}_{\left[e^t\right]_0^1 = e-1}\right) =$$

$$= \frac{1}{2}(e-(e-1)) = \frac{1}{2}\Big] = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta \cos\varphi\, d\varphi = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \underbrace{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}}_{\frac{\pi}{2}-0-0+0} \cos\varphi\, d\varphi = \frac{\pi}{4}\underbrace{\left[\sin\varphi\right]_{-\pi/2}^{\pi/2}}_{=1+1} = \underline{\frac{\pi}{2}}$$

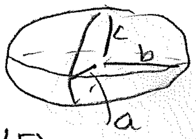
6.26 c) $0 \le x \le y$  ger $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$

$D: \begin{cases} x^2+y^2+z^2 \le 3 \\ 0 \le x \le y \end{cases}$ ger $\begin{cases} 0 \le r \le \sqrt{3} \\ 0 \le \theta \le \pi \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \end{cases}$ $\Rightarrow$

$$\iiint_D x\,dx\,dy\,dz = \int_{\pi/4}^{\pi/2}\int_0^{\pi}\int_0^{\sqrt{3}} r\sin\theta\cos\varphi \cdot \underbrace{r^2\sin\theta}_{\text{determinant}}\, dr\, d\theta\, d\varphi = \int_{\pi/4}^{\pi/2}\int_0^{\pi} \underbrace{\left[\frac{r^4}{4}\right]_0^{\sqrt{3}}}_{=9/4} \frac{1-\cos 2\theta}{2}\cos\varphi\, d\theta\, d\varphi =$$

$$= \frac{9}{4}\int_{\pi/4}^{\pi/2} \underbrace{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}}_{=\pi/2} \cos\varphi\, d\varphi = \frac{9\pi}{8}\left[\sin\varphi\right]_{\pi/4}^{\pi/2} = \underline{\frac{9\pi}{8}\left(1 - \frac{1}{\sqrt{2}}\right)} = \underline{\frac{9\pi}{16}(2-\sqrt{2})}$$

6.30 a) Ellipsoid D: $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1$ (fylld)



$a,b,c>0$

$u=\frac{x}{a},\ v=\frac{y}{b},\ w=\frac{z}{c} \Rightarrow u^2+v^2+w^2\le 1$ klot, radie 1 (E)

$x=au,\ y=bv,\ z=cw \quad \frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \quad \Rightarrow$

$$\text{Volym}(D) = \iiint_D 1\,dxdydz = \iiint_E abc\,dudvdw = abc \underbrace{\iiint_E dudvdw}_{\text{volym (enhetsklot)}} = \underline{abc\cdot\frac{4\pi}{3}}$$

6.30 b) D: $3x^2+2y^2+z^2+2xz-2yz\le 1$

$$z^2+2xz-2yz+2y^2+3x^2 = (z+x-y)^2+y^2+2x^2+2xy = (z+x-y)^2+(y+x)^2+x^2$$

Sätt $\begin{cases} u = z+x-y \\ v = y+x \\ w = x \end{cases} \Rightarrow$ nytt område E: $u^2+v^2+w^2\le 1$, enhetsklot, volym(E) $=\frac{4\pi}{3}$

Determinant $\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \Rightarrow \frac{d(x,y,z)}{d(u,v,w)} = \frac{1}{-1} = -1$

$$\text{volym}(D) = \iiint_D 1\,dxdydz = \iiint_E \underbrace{\left|\frac{d(x,y,z)}{d(u,v,w)}\right|}_{=|-1|=1} dudvdw = \iiint_E dudvdw = \text{volym}(E) = \underline{\frac{4\pi}{3}}$$