2.37 Polära koordinater $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$

a) $\begin{cases} u'_\rho = u'_x x'_\rho + u'_y y'_\rho = \cos\varphi\, u'_x + \sin\varphi\, u'_y & (1) \\ u'_\varphi = u'_x x'_\varphi + u'_y y'_\varphi = -\rho\sin\varphi\, u'_x + \rho\cos\varphi\, u'_y & (2) \end{cases}$ (*)

b) För att lösa ut u'_x ur (*), ta $\rho\cos\varphi\cdot(1) - \sin\varphi\cdot(2)$, då går u'_y bort:

$$\rho\cos\varphi\, u'_\rho - \sin\varphi\, u'_\varphi = \rho\cos\varphi(\cos\varphi\, u'_x + \sin\varphi\, u'_y) - \sin\varphi(-\rho\sin\varphi\, u'_x + \rho\cos\varphi\, u'_y) =$$

$$= \rho(\cos^2\varphi + \sin^2\varphi)u'_x + 0u'_y = \rho u'_x$$

$$\Rightarrow u'_x = \cos\varphi\, u'_\rho - \frac{\sin\varphi}{\rho}u'_\varphi$$

Likadant leder $\rho\sin\varphi\cdot(1) + \cos\varphi\cdot(2)$ till $u'_y = \sin\varphi\, u'_\rho + \frac{\cos\varphi}{\rho}u'_\varphi$

c) Transformera Laplaces ekv. $u_{xx} + u_{yy} = 0$ till polära koordinater

Från b): $(\ )'_x = \cos\varphi(\ )'_\rho - \frac{\sin\varphi}{\rho}(\ )'_\varphi$, $(\ )'_y = \sin\varphi(\ )'_\rho + \frac{\cos\varphi}{\rho}(\ )'_\varphi$ $\Rightarrow$

$$u''_{xx} = (u'_x)'_x = \left(\cos\varphi\, u'_\rho - \frac{\sin\varphi}{\rho}u'_\varphi\right)'_x = \cos\varphi\Big(\underbrace{\cos\varphi}_{\text{"konstant"}} u'_\rho - \underbrace{\frac{\sin\varphi}{\rho}u'_\varphi}_{\text{produkt}}\Big)'_\rho - \frac{\sin\varphi}{\rho}\Big(\underbrace{\cos\varphi\, u'_\rho}_{\text{prod.}} - \underbrace{\frac{\sin\varphi}{\rho}u'_\varphi}_{\text{prod.}}\Big)'_\varphi$$

$$= \cos\varphi\left(\cos\varphi\, u''_{\rho\rho} + \frac{\sin\varphi}{\rho^2}u'_\varphi - \frac{\sin\varphi}{\rho}u''_{\rho\varphi}\right) - \frac{\sin\varphi}{\rho}\left(-\sin\varphi\, u'_\rho + \cos\varphi\, u''_{\rho\varphi} - \frac{\cos\varphi}{\rho}u'_\varphi - \frac{\sin\varphi}{\rho}u''_{\varphi\varphi}\right) =$$

$$= \cos^2\varphi\, u''_{\rho\rho} + \frac{2\cos\varphi\sin\varphi}{\rho^2}u'_\varphi - \frac{2\sin\varphi\cos\varphi}{\rho}u''_{\rho\varphi} + \frac{\sin^2\varphi}{\rho}u'_\rho + \frac{\sin^2\varphi}{\rho^2}u''_{\varphi\varphi}$$

På samma sätt beräknas $u''_{yy} = \sin^2\varphi\, u''_{\rho\rho} - \frac{2\cos\varphi\sin\varphi}{\rho^2}u'_\varphi + \frac{2\cos\varphi\sin\varphi}{\rho}u''_{\rho\varphi} + \frac{\cos^2\varphi}{\rho}u'_\rho + \frac{\cos^2\varphi}{\rho^2}u''_{\varphi\varphi}$

$$\Rightarrow u''_{xx} + u''_{yy} = /\sin^2\varphi + \cos^2\varphi = 1/ = u''_{\rho\rho} + \frac{1}{\rho}u'_\rho + \frac{1}{\rho^2}u''_{\varphi\varphi}$$

d) Vinkeloberoende lösningar på Laplaces ekv.: $u'_\varphi = 0$, $u''_{\varphi\varphi} = 0$ $\Rightarrow$

$u''_{\rho\rho} + \frac{1}{\rho}u'_\rho = 0$. Sätt $v(\rho) = u'_\rho$, v envariabelfunktion (ty ober. av φ) $\Rightarrow v' + \frac{1}{\rho}v = 0$

[linjär ekv. ordn 1, mult. med integrerande faktor $e^{\int\frac{1}{\rho}d\rho} = e^{\ln\rho} = \rho$] $\Rightarrow$

$$\rho v' + v = 0 \underset{\text{produkt-regel}}{\Rightarrow} (\rho v)' = 0 \Rightarrow \rho v = \underbrace{c}_{\text{konstant}} \Rightarrow v = \frac{c}{\rho} \Rightarrow$$

$$u'_\rho = \frac{c}{\rho} \Rightarrow u = c\ln\rho + \underbrace{k}_{\substack{\text{konstant} \\ \text{(ty oberoav }\varphi)}} = c\ln\sqrt{x^2+y^2} + k = \frac{c}{2}\ln(x^2+y^2) + k$$