Lösningar 2.70 dg

2.70d) $f(x,y) = x+y-3\ln(2+xy)$, $x>0, y>0$

Sök stationära punkter $\begin{cases} f'_x = 1-3\dfrac{y}{2+xy} = 0 & (1) \\ f'_y = 1-3\dfrac{x}{2+xy} = 0 & (2) \end{cases}$ (1)-(2) ger $\dfrac{-3y+3x}{2+xy} = 0 \Rightarrow -3y+3x=0 \Rightarrow y=x$

$y=x$ i (1) $\Rightarrow 1-\dfrac{3x}{2+x^2}=0 \Rightarrow 2+x^2-3x=0 \Rightarrow x=\frac{3}{2}\pm\sqrt{\frac{9}{4}-2} = \frac{3}{2}\pm\frac{1}{2} \Rightarrow x_1=2, x_2=1$

$y=x \Rightarrow y_1=2, y_2=1 \Rightarrow$ 2 stationära punkter $(x_1,y_1)=(2,2)$ och $(x_2,y_2)=(1,1)$

$f''_{xx}(x,y) = \dfrac{3y^2}{(2+xy)^2}$, $f''_{xy}(x,y) \underset{\substack{\uparrow \\ \text{kvotregeln}}}{=} -3\dfrac{1\cdot(2+xy)-x\cdot y}{(2+xy)^2} = \dfrac{-6}{(2+xy)^2}$, $f''_{yy}(x,y) = \dfrac{3x^2}{(2+xy)^2}$

Punkt $(x_1,y_1)=(2,2)$: $f''_{xx} = \frac{12}{6^2}=\frac{1}{3}$, $f''_{xy} = \frac{-6}{6^2}=-\frac{1}{6}$, $f''_{yy} = \frac{12}{6^2}=\frac{1}{3} \Rightarrow$

Alternativ 1 $\quad Hf=\begin{pmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{pmatrix}$, egenvärden $\begin{vmatrix} 1/3-\lambda & -1/6 \\ -1/6 & 1/3-\lambda \end{vmatrix}=0 \Rightarrow (\lambda-\frac{1}{3})^2=(\frac{1}{6})^2 \Rightarrow \lambda-\frac{1}{3}=\pm\frac{1}{6}$

$\Rightarrow \lambda_1=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}>0, \lambda_2=\frac{1}{3}-\frac{1}{6}=\frac{1}{6}>0$

$\lambda_1>0, \lambda_2>0 \Rightarrow$ positivt definit $\Rightarrow$ lokalt min

Alt 2 $\quad Q(h,k)=\frac{1}{3}h^2-\frac{1}{6}\cdot 2hk+\frac{1}{3}k^2=\frac{1}{3}(h^2-hk+k^2)=\frac{1}{3}\Big[\underbrace{(h-\tfrac{k}{2})^2}_{+}+\underbrace{\tfrac{3}{4}k^2}_{+}\Big]$, + + $\Rightarrow$ pos. definit $\Rightarrow$ lokalt min

Punkt $(x_2,y_2)=(1,1)$: $f''_{xx}=\frac{3}{3^2}=\frac{1}{3}$, $f''_{xy}=\frac{-6}{3^2}=-\frac{2}{3}$, $f''_{yy}=\frac{3}{3^2}=\frac{1}{3} \Rightarrow$

Alt 1 $\quad \begin{vmatrix} 1/3-\lambda & -2/3 \\ -2/3 & 1/3-\lambda \end{vmatrix}=(\frac{1}{3}-\lambda)^2-\frac{4}{9}=0 \Rightarrow \lambda-\frac{1}{3}=\pm\frac{2}{3} \Rightarrow \lambda_1=1>0, \lambda_2=-\frac{1}{3}<0$ $\quad + \; - \Rightarrow$ indefinit $\Rightarrow$ sadelpunkt

Alt 2 $\quad Q(h,k)=\frac{1}{3}h^2-\frac{2}{3}\cdot 2hk+\frac{1}{3}k^2=\frac{1}{3}(h^2-4hk+k^2)=\frac{1}{3}\Big[\underbrace{(h-2k)^2}_{+}-\underbrace{3k^2}_{-}\Big]$, + − $\Rightarrow$ indefinit $\Rightarrow$ sadelpunkt

Svar: (2,2) är lokalt min, (1,1) är en sadelpunkt

2.70g) $f(x,y)=x^3+3xy^2-15x-12y$ $\quad \begin{cases} f'_x=3x^2+3y^2-15=0 \\ f'_y=6xy-12=0 \end{cases} \Rightarrow \begin{cases} x^2+y^2=5 \\ xy=2 \Rightarrow y=\frac{2}{x} \end{cases} \Rightarrow x^2+\frac{4}{x^2}=5$

[likt exempel på föreläsning 1!]

$\Rightarrow (x^2)^2-5x^2+4=0 \Rightarrow x^2=\frac{5}{2}\pm\sqrt{\frac{25}{4}-\frac{16}{4}}=\frac{5\pm 3}{2} \Rightarrow x^2=4$ eller $x^2=1 \Rightarrow x_1=2, x_2=-2,$

$x_3=1, x_4=-1$ och $y=\frac{2}{x}$ ger $y_1=1, y_2=-1, y_3=2, y_4=-2$

4 stationära punkter:

	(2,1)	(−2,−1)	(1,2)	(−1,−2)
$f''_{xx}=6x$:	12	−12	6	−6
$f''_{xy}=6y$:	6	−6	12	−12
$f''_{yy}=6x$:	12	−12	6	−6

Kvadratkomplettering $\Rightarrow$

(2,1) : $Q(h,k)=12h^2+6\cdot 2hk+12k^2=12(h^2+hk+k^2)=12\big[(h+\frac{k}{2})^2+\frac{3k^2}{4}\big]$, + + $\Rightarrow$ Q pos. def. $\Rightarrow$ lok. min

(−2,−1) : $Q(h,k)=-12h^2-6\cdot 2hk-12k^2=-12\big[(h+\frac{k}{2})^2+\frac{3k^2}{4}\big]$, − − $\Rightarrow$ Q neg. def. $\Rightarrow$ lok. max

(1,2) : $Q(h,k)=6h^2+12\cdot 2hk+6k^2=6(h^2+4hk+k^2)=6\big[(h+2k)^2-3k^2\big]$, + − $\Rightarrow$ Q indefinit $\Rightarrow$ sadelp.

(−1,−2) : $Q(h,k)=-6h^2-12\cdot 2hk-6k^2=-6\big[(h+2k)^2-3k^2\big]$, − + $\Rightarrow$ Q indefinit $\Rightarrow$ sadelpunkt

Svar: Lokalt min i (2,1), lokalt max i (−2,−1), sadelpunkter i (1,2) och (−1,−2)