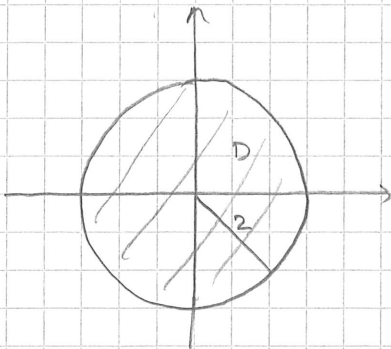


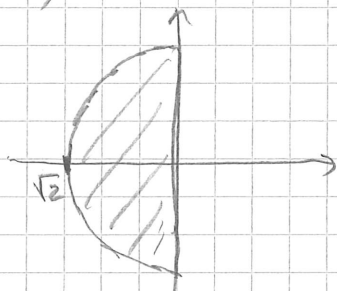
10.1)

a)



$$\iint_D e^{x^2+y^2} dx dy = \int_0^{2\pi} \left(\int_0^2 e^{\rho^2} \rho d\rho \right) d\varphi = 2\pi \left[\frac{e^{\rho^2}}{2} \right]_0^2 = \underline{\underline{\pi(e^4-1)}}.$$

b)



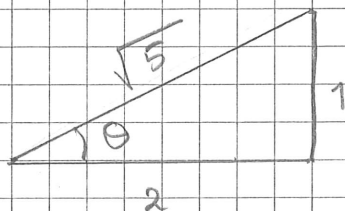
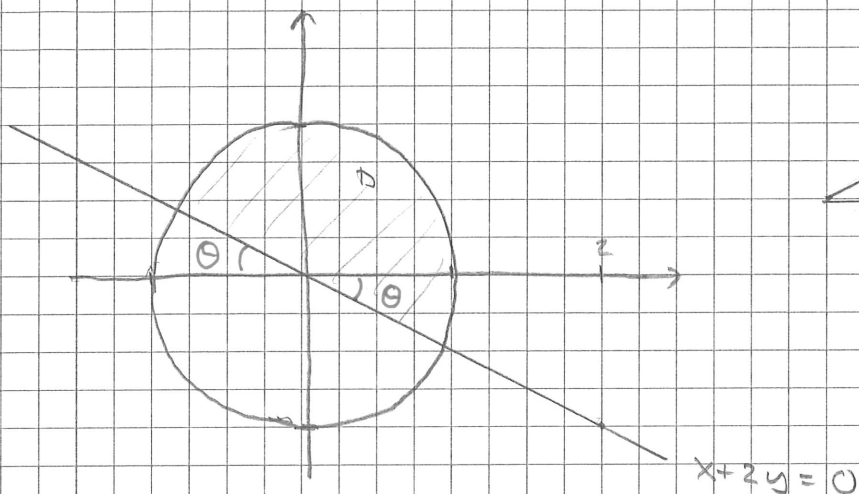
$$\pi/2 \leq \varphi \leq 3\pi/2 \quad 0 \leq \rho \leq \sqrt{2}$$

$$\iint_D \frac{x}{1+(x^2+y^2)^{3/2}} dx dy = \int_{\pi/2}^{3\pi/2} \left(\int_0^{\sqrt{2}} \frac{\rho \cos \varphi}{1+\rho^3} \rho d\rho \right) d\varphi =$$

$$= \int_{\pi/2}^{3\pi/2} \cos \varphi d\varphi \cdot \int_0^{\sqrt{2}} \frac{\rho^2}{1+\rho^3} d\rho = \left[\sin \varphi \right]_{\pi/2}^{3\pi/2} \cdot \left[\frac{1}{3} \ln(1+\rho^3) \right]_0^{\sqrt{2}} =$$

$$= \underline{\underline{-\frac{2}{3} \ln(1+2\sqrt{2})}}$$

10.11c) Area



$$\iint_D (x+2y) dx dy = \int_{-\theta}^{\pi-\theta} \left(\int_0^1 (p \cos \varphi + 2p \sin \varphi) p dp \right) d\varphi$$

$$= \int_{-\theta}^{\pi-\theta} (\cos \varphi + 2 \sin \varphi) d\varphi \cdot \int_0^1 p^2 dp =$$

$$= \left[\sin \varphi - 2 \cos \varphi \right]_{-\theta}^{\pi-\theta} \cdot \left[\frac{p^3}{3} \right]_0^1 =$$

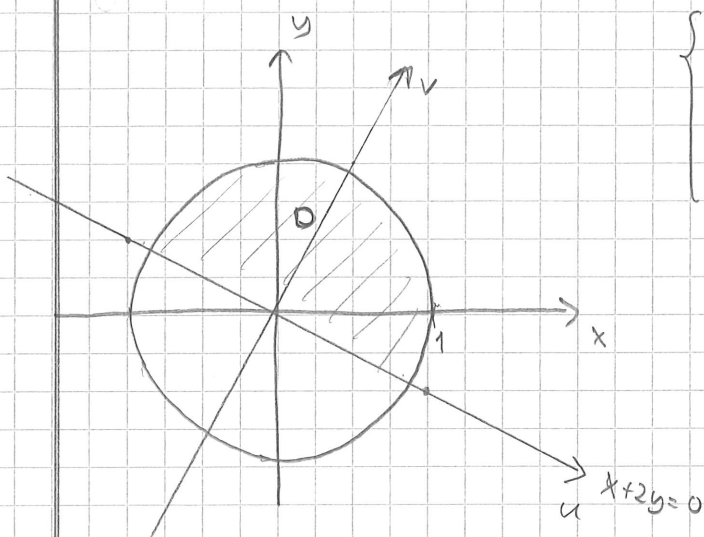
$$= \left(\underbrace{\sin(\pi-\theta)}_{=\sin \theta} - 2 \underbrace{\cos(\pi-\theta)}_{=-\cos \theta} - \underbrace{\sin(-\theta)}_{=-\sin \theta} + 2 \underbrace{\cos(-\theta)}_{=\cos \theta} \right) \frac{1}{3}$$

$$= \frac{1}{3} (2 \sin \theta + 4 \cos \theta) =$$

/ Från triangeln ovan får vi $\sin \theta = \frac{1}{\sqrt{5}}$, $\cos \theta = \frac{2}{\sqrt{5}}$ /

$$= \frac{1}{3} \left(\frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} \right) = \frac{10}{3\sqrt{5}} = \underline{\underline{\frac{2\sqrt{5}}{3}}}$$

10.1c), Alternativ lösning med ON-basbyte.



$$\begin{cases} u = \frac{2x-y}{\sqrt{5}} \\ v = \frac{x+2y}{\sqrt{5}} \end{cases}$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 1 & 2 \end{vmatrix} = 1.$$

$$x^2 + y^2 = u^2 + v^2, \text{ så}$$

$$\Omega = \{(u,v) : u^2 + v^2 \leq 1, v \geq 0\}$$

$$\iint_D (x+2y) dx dy = \iint_{\Omega} \sqrt{5} v \left| \frac{d(x,y)}{d(u,v)} \right| du dv =$$

$$= \int_0^{\pi} \left(\int_0^1 \sqrt{5} \rho \sin \varphi \rho d\rho \right) d\varphi = \sqrt{5} \int_0^{\pi} \sin \varphi d\varphi \cdot \int_0^1 \rho^2 d\rho = \underline{\underline{\frac{2\sqrt{5}}{3}}}$$

d)

$$\iint_D |x+y| dx dy = \left/ \begin{array}{l} u = \frac{x+y}{\sqrt{2}} \\ v = \frac{-x+y}{\sqrt{2}} \end{array} \right. \frac{d(u,v)}{d(x,y)} = \frac{1}{\sqrt{2}^2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 \left/ \right.$$

$D \mapsto \Omega = \{(u,v) : u^2 + v^2 \leq 1\}$

$$= \iint_{\Omega} |\sqrt{2}u| \left| \frac{d(x,y)}{d(u,v)} \right| du dv = \sqrt{2} \int_0^{2\pi} \left(\int_0^1 |\rho \cos \varphi| \rho d\rho \right) d\varphi =$$

$$= \sqrt{2} \int_0^{2\pi} |\cos \varphi| \cdot \int_0^1 \rho^2 d\rho = 4\sqrt{2} \cdot \int_0^{\pi/2} \cos \varphi d\varphi \cdot \frac{1}{3} = \underline{\underline{\frac{4\sqrt{2}}{3}}}.$$

10.2)

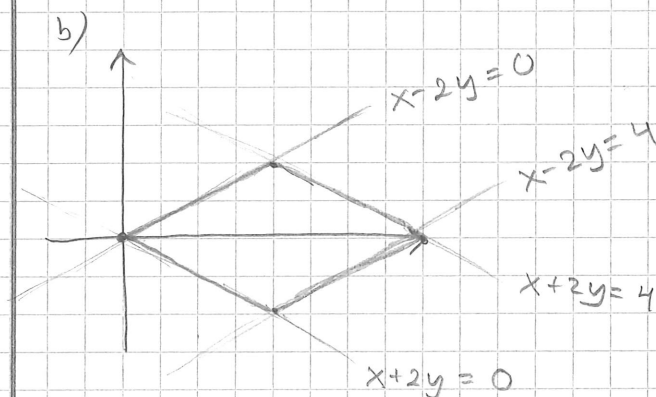
a) $D : 0 \leq x+2y \leq 1, -1 \leq 2x+y \leq 1.$

Med $u = x+2y, v = 2x+y$ ges området av
 $0 \leq u \leq 1, -1 \leq v \leq 1.$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\iint_D (x+2y) \cos(2x+y) dx dy = \int_0^1 \left(\int_{-1}^1 u \cos v \left| \frac{d(x,y)}{d(u,v)} \right| dv \right) du =$$

$$= \frac{1}{3} \int_0^1 u du \int_{-1}^1 \cos v dv = \frac{1}{3} \cdot \frac{1}{2} \cdot (\sin 1 - \sin(-1)) = \underline{\underline{\frac{\sin 1}{3}}}$$



$$\begin{cases} u = x+2y \\ v = x-2y \end{cases} \quad \text{Området ges av } 0 \leq u \leq 4, 0 \leq v \leq 4.$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4.$$

$$\iint_D (x+2y) \exp(x-2y) dx dy = \int_0^4 \left(\int_0^4 u \exp v \left| \frac{d(x,y)}{d(u,v)} \right| du \right) dv$$

$$= \frac{1}{4} \int_0^4 u du \int_0^4 \exp v dv = \underline{\underline{\frac{2(e^4-1)}{4}}}$$

10.2)

c) Med $u = x + y$, $v = 2x - 4y$ ges området av
 $1 \leq u \leq v \leq 3$

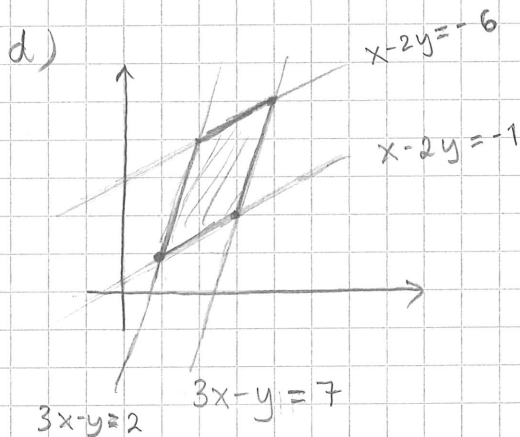
$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} = -6$$

$$\iint_D \frac{x-y+1}{x+y} dx dy = \left/ \begin{matrix} u+v=3x-3y \\ dx dy = \frac{1}{6} du dv \end{matrix} \right/ = \frac{1}{6} \int_1^3 \left(\int_1^v \frac{\frac{u+v}{3} + 1}{u} du \right) dv$$

$$= \frac{1}{18} \int_1^3 \left(\int_1^v \left(1 + \frac{v+3}{u} \right) du \right) dv = \frac{1}{18} \int_1^3 \left[u + (v+3) \ln u \right]_{u=1}^v dv$$

$$= \frac{1}{18} \int_1^3 (v + (v+3) \ln v - 1) dv = \frac{1}{18} \left[\frac{v^2}{2} + \frac{v^2}{2} \ln v - \frac{v^2}{4} + 3v \ln v - 3v - v \right]_1^3$$

$$= \underline{\underline{\frac{3 \ln 3}{4} - \frac{1}{3}}}$$



$$u = x - 2y, v = 3x - y$$

Området ges av

$$-6 \leq u \leq -1, 2 \leq v \leq 7.$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$$

$$dx dy = \frac{1}{5} du dv$$

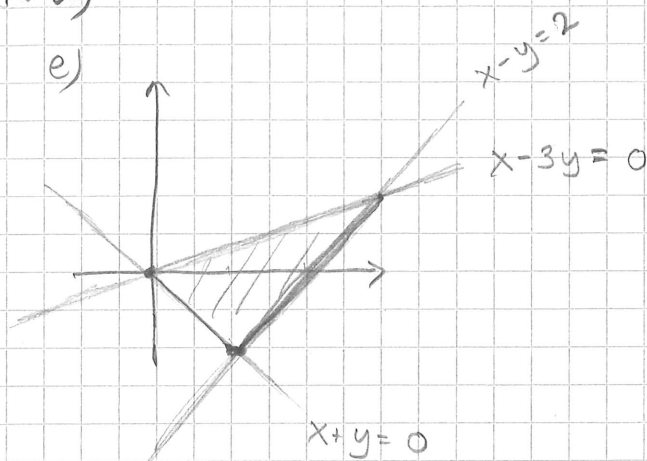
$$\iint_D x^2 dx dy = \left/ \begin{matrix} 2v-u=6x-2y-x+2y=5x \\ dx dy = \frac{1}{5} du dv \end{matrix} \right/ = \frac{1}{5} \int_2^7 \left(\int_{-6}^{-1} \left(\frac{2v-u}{5} \right)^2 du \right) dv$$

$$= \frac{1}{5^3} \int_2^7 \left(\int_{-6}^{-1} (4v^2 - 4uv + u^2) du \right) dv = \frac{1}{5^3} \int_2^7 \left[4v^2 u - 2u^2 v + \frac{u^3}{3} \right]_{-6}^{-1} dv$$

$$= \frac{1}{5^3} \int_2^7 \left(-4v^2 - 2v - \frac{1}{3} + 24v^2 + 72v + 72 \right) dv = \dots = \frac{100}{3}$$

10.2)

e)

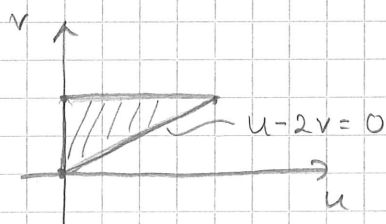


Integranden $\frac{1}{(1+x^2-y^2)^2} = \frac{1}{(1+(x+y)(x-y))^2}$

använder tillsammans med bilden ovan att

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \text{ är lämpligt byte.}$$

I uv -planet för punkterna som i xy -planet har koordinater $(0,0)$, $(1,-1)$ och $(3,1)$ koord. $(0,0)$, $(0,2)$, $(4,2)$



$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

$$\begin{aligned} \iint_D \frac{1}{(1+x^2-y^2)^2} dx dy &= \int_0^2 \left(\int_0^{2v} \frac{1}{(1+uv)^2} \frac{1}{2} du \right) dv \\ &= \frac{1}{2} \int_0^2 \left[\frac{-1}{v(1+uv)} \right]_0^{2v} dv = \frac{1}{2} \int_0^2 \left(\frac{1}{v} - \frac{1}{v(1+2v^2)} \right) dv \\ &= \frac{1}{2} \int_0^2 \left(\frac{1}{v} - \frac{1}{v} + \frac{2v}{1+2v^2} \right) dv = \frac{1}{4} \left[\ln(1+2v^2) \right]_0^2 = \\ &= \frac{1}{4} \ln(9) = \underline{\underline{\frac{\ln 3}{2}}} \end{aligned}$$

10.3)

a)

Med $x = 2p \cos \varphi$, $y = 3p \sin \varphi$ får vi

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4p^2 \cos^2 \varphi}{4} + \frac{9p^2 \sin^2 \varphi}{9} = p^2 \leq 1.$$

Så i $p\varphi$ -planet ges området av $0 \leq \varphi \leq 2\pi$,

$$0 \leq p \leq 1.$$

$$\frac{d(x,y)}{d(p,\varphi)} = \begin{vmatrix} 2 \cos \varphi & -2p \sin \varphi \\ 3 \sin \varphi & 3p \cos \varphi \end{vmatrix} = 6p$$

$$\text{D.v.s. } dx dy = 6p dp d\varphi.$$

$$\begin{aligned} \iint_D (x^2 + y^2) dx dy &= \int_0^{2\pi} \left(\int_0^1 (4p^2 \cos^2 \varphi + 9p^2 \sin^2 \varphi) 6p dp \right) d\varphi = \\ &= 6 \int_0^{2\pi} (4 \cos^2 \varphi + 9 \sin^2 \varphi) d\varphi \cdot \int_0^1 p^3 dp = \dots = \frac{39\pi}{2}. \end{aligned}$$

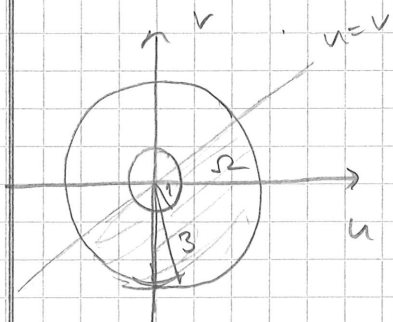
b)

Om vi först byter koordinater $\begin{cases} u = x \\ v = 3y \end{cases}$

får vi $dx dy = \frac{1}{3} du dv$, och att

området ges av $1 \leq u^2 + v^2 \leq 9$, $\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$.

$$\begin{aligned} \iint_D x^3 dx dy &= \iint_{\Omega} u^3 \frac{1}{3} du dv = \frac{1}{3} \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \left(\int_1^3 p^3 \cos^3 \varphi p dp \right) d\varphi \\ &= \frac{1}{3} \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos^3 \varphi d\varphi \cdot \int_1^3 p^4 dp = \\ &= \frac{1}{3} \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos \varphi (1 - \sin^2 \varphi) d\varphi \cdot \left[\frac{p^5}{5} \right]_1^3 \\ &= \frac{242}{15} \left[\sin \varphi - \frac{\sin^3 \varphi}{3} \right]_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{242}{15} \left(\frac{2}{\sqrt{2}} - \frac{2}{2\sqrt{2} \cdot 3} \right) = \frac{121\sqrt{2}}{9} \end{aligned}$$



I pol. koord. ges Ω
av $-\frac{3\pi}{4} \leq \varphi \leq \frac{\pi}{4}$, $1 \leq p \leq 3$

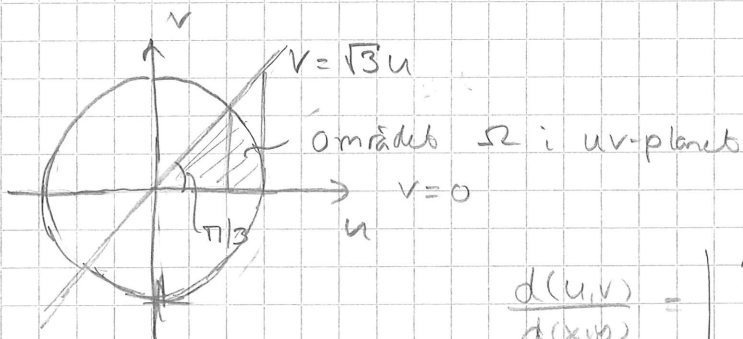
10.3)

c)

$$x^2 + 2xy + 4y^2 = (x+y)^2 + 3y^2 \leq 1 \quad x \geq 0, y \geq 0$$

$$\begin{cases} u = x+y \\ v = \sqrt{3}y \end{cases} \Leftrightarrow \begin{cases} x+y=u \\ y=\frac{1}{\sqrt{3}}v \end{cases} \Leftrightarrow \begin{cases} x = u - \frac{1}{\sqrt{3}}v \\ y = \frac{1}{\sqrt{3}}v \end{cases}$$

! uv-planets ges omrdet av $u^2 + v^2 \leq 1$, $u - \frac{1}{\sqrt{3}}v \geq 0$,
 $\frac{1}{\sqrt{3}}v \geq 0$

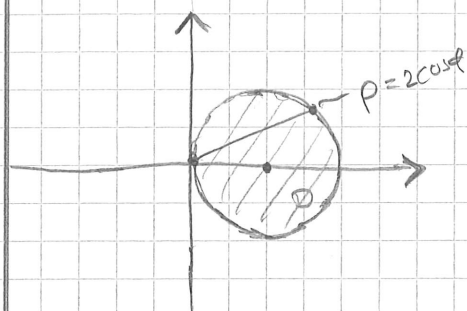


$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = \sqrt{3}$$

$$\begin{aligned} \iint_D x \, dx \, dy &= \iint_{\Omega} \left(u - \frac{v}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} \, du \, dv = \\ &= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \left(\int_0^1 \left(p \cos \varphi - \frac{p \sin \varphi}{\sqrt{3}}\right) p \, dp \right) d\varphi = \\ &= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \left(\cos \varphi - \frac{\sin \varphi}{\sqrt{3}} \right) d\varphi \cdot \int_0^1 p^2 \, dp = \frac{1}{3\sqrt{3}} \left[\sin \varphi + \frac{\cos \varphi}{\sqrt{3}} \right]_0^{\pi/3} = \\ &= \frac{1}{3\sqrt{3}} \left(\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \right) = \frac{1}{6} - \frac{1}{18} = \underline{\underline{\frac{1}{9}}} \end{aligned}$$

10.4)

$$x^2 + y^2 \leq 2x \Leftrightarrow (x-1)^2 + y^2 \leq 1$$



| polära koordinater

$$x = p \cos \varphi$$

$$y = p \sin \varphi \quad \text{ges } D \text{ av}$$

$$(p \cos \varphi - 1)^2 + p^2 \sin^2 \varphi =$$

$$= p^2 - 2p \cos \varphi + 1 \leq 1$$

$$\Leftrightarrow 0 \leq p \leq 2 \cos \varphi, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

Om vi å andra sidan gör ett linjärt variabelbyte

$u = x - 1, v = y$ får vi vanliga enhetscirkeln i uv -planet

$$; u^2 + v^2 \leq 1.$$

$$a) \iint_D (x^2 + y^2) dx dy = \left/ \begin{array}{l} x-1=u \\ y=v \end{array} \right. \Leftrightarrow \begin{array}{l} x=1+u \\ y=v \end{array} \left/ \begin{array}{l} dx dy = du dv \end{array} \right. =$$

$$= \iint_{u^2+v^2 \leq 1} ((1+u)^2 + v^2) du dv = \int_0^{2\pi} \left(\int_0^1 (p^2 + 2p \cos \varphi + 1) p dp \right) d\varphi$$

$$= \int_0^{2\pi} \left[\frac{p^4}{4} + \frac{2p^3}{3} \cos \varphi + \frac{p^2}{2} \right]_{p=0}^1 d\varphi = \int_0^{2\pi} \left(\frac{1}{4} + \frac{2 \cos \varphi}{3} + \frac{1}{2} \right) d\varphi$$

$$= \left[\frac{3\varphi}{4} + \frac{2 \sin \varphi}{3} \right]_0^{2\pi} = \underline{\underline{\frac{3\pi}{2}}}$$

$$b) \iint_D \sqrt{x^2 + y^2} dx dy = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2 \cos \varphi} p \cdot p dp \right) d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{p^3}{3} \right]_0^{2 \cos \varphi} d\varphi = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \varphi d\varphi = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos \varphi (1 - \sin^2 \varphi) d\varphi =$$

$$= \frac{8}{3} \left[\sin \varphi - \frac{\sin^3 \varphi}{3} \right]_{-\pi/2}^{\pi/2} = \frac{8}{3} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \underline{\underline{\frac{32}{9}}}$$

$$10.5) \quad 1 \leq xy \leq 2, \quad 0 \leq x \leq y \leq 2x$$

$$\Rightarrow \quad 1 \leq xy \leq 2, \quad 0 \leq x, \quad xy \leq y^2 \leq 2xy$$

Med $u = xy$, $v = y^2$ avbildas D injektivt på området $1 \leq u \leq 2$, $u \leq v \leq 2u$.

OBS! Valet av dessa koordinater är anpassat dels efter området, och dels efter integranden

$$y^2 \sin y^2.$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} y & x \\ 0 & 2y \end{vmatrix} = 2y^2 = 2v, \quad dx dy = \frac{1}{2v} du dv.$$

$$\iint_D y^2 \sin y^2 dx dy = \int_1^2 \left(\int_u^{2u} v \sin v \frac{1}{2v} dv \right) du =$$

$$= \frac{1}{2} \int_1^2 \left[-\cos v \right]_{v=u}^{2u} dv = \frac{1}{2} \int_1^2 (\cos u - \cos 2u) du =$$

$$= \frac{1}{2} \left[\sin u - \frac{\sin 2u}{2} \right]_1^2 = \frac{1}{2} \left(\sin 2 - \frac{\sin 4}{2} - \sin 1 + \frac{\sin 2}{2} \right) =$$

$$= \frac{3 \sin 2 - \sin 4 - 2 \sin 1}{4}$$