

11.1)

$$D: 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 4.$$

D har volum $1 \cdot 2 \cdot 4 = 8$.

$$\sqrt[3]{x} + y + \sqrt{z} \leq \sqrt[3]{1} + 2 + \sqrt{4} = 5$$

$$I \leq 8 \cdot 5 = 40.$$

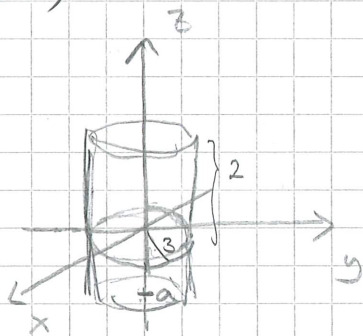
$$\int_0^1 \left(\int_0^2 \left(\int_0^4 (\sqrt[3]{x} + y + \sqrt{z}) dz \right) dy \right) dx =$$

$$= 8 \int_0^1 \sqrt[3]{x} dx + 4 \int_0^2 y dy + 2 \int_0^4 \sqrt{z} dz =$$

$$= 8 \left[\frac{3}{4} x^{4/3} \right]_0^1 + 4 \left[y^2/2 \right]_0^2 + 2 \left[\frac{2}{3} z^{3/2} \right]_0^4 =$$

$$= 6 + 8 + \frac{32}{3} = \frac{74}{3}$$

11.2)



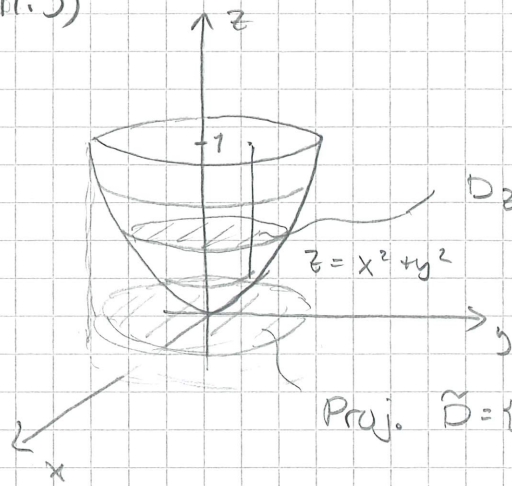
$$D = \{(x, y, z) : x^2 + y^2 \leq 9, a \leq z \leq 2+a\}$$

$$\iiint_D (x^2 + y^2) z \, dx \, dy \, dz = \iint_{x^2 + y^2 \leq 9} \left(\int_a^{2+a} (x^2 + y^2) z \, dz \right) dx \, dy$$

$$= \int_0^{2\pi} \int_0^3 \rho^2 \rho \, d\rho \, d\phi \cdot \int_a^{2+a} z \, dz = 2\pi \left[\frac{\rho^4}{4} \right]_0^3 \cdot \left[\frac{z^2}{2} \right]_a^{2+a} =$$

$$= \frac{81\pi}{4} ((2+a)^2 - a^2) = 81\pi(1+a).$$

11.3)



$$D_z = \{(x, y) : x^2 + y^2 \leq z\}, \quad a=0, b=1.$$

$$z = x^2 + y^2$$

$$\text{Proj. } \tilde{D} = \{(x, y) : x^2 + y^2 \leq 1\}$$

$$A(x, y) = x^2 + y^2, \quad B(x, y) = 1$$

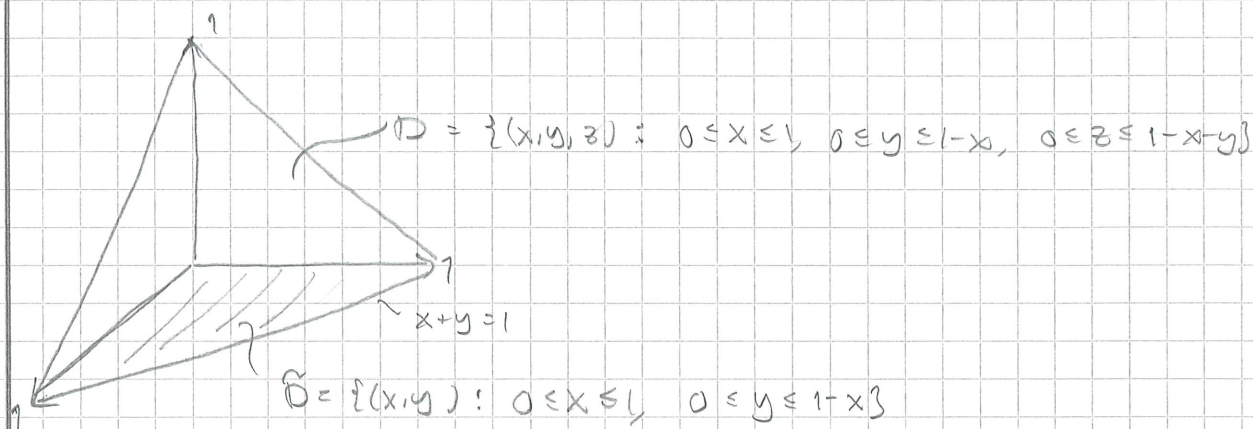
$$\iiint_D z \sqrt{x^2 + y^2} \, dx \, dy \, dz = \iint_{\tilde{D}} \left(\int_{x^2+y^2}^1 z \sqrt{x^2 + y^2} \, dz \right) dx \, dy =$$

$$= \iint_{\tilde{D}} \sqrt{x^2 + y^2} \left(\frac{1}{2} - \frac{(x^2 + y^2)^2}{2} \right) dx \, dy =$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_0^1 \rho^2 (1 - \rho^4) \rho \, d\rho \right) d\varphi =$$

$$= \pi \int_0^1 (\rho^2 - \rho^6) d\rho = \pi \left[\frac{\rho^3}{3} - \frac{\rho^7}{7} \right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{7} \right) = \underline{\underline{\frac{4\pi}{21}}}$$

11.4)



$$\iiint_D \frac{dx dy dz}{(1+x+y+z)^3} = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz \right) dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2} \right]_{z=0}^{1-x-y} dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\frac{-1}{2 \cdot 2^2} + \frac{1}{2(1+x+y)^2} \right) dy \right) dx =$$

$$= \int_0^1 \left[\frac{-y}{8} - \frac{1}{2(1+x+y)} \right]_{y=0}^{1-x} dx =$$

$$= \int_0^1 \left(\frac{-(1-x)}{8} - \frac{1}{2 \cdot 2} + \frac{1}{2(1+x)} \right) dx =$$

$$= \left[\frac{x^2}{16} - \frac{3x}{8} + \frac{1}{2} \ln(1+x) \right]_0^1 = \frac{1}{16} - \frac{3}{8} + \frac{1}{2} \ln 2 =$$

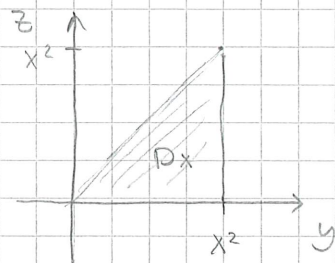
$$= \ln 2 - \frac{5}{16}$$

11.5)

$$D = \{(x, y, z) : 0 \leq z \leq y \leq x^2 \leq 1\}$$

$$\begin{aligned} a) D &= \{(x, y, z) : -1 \leq x^2 \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq y\} \\ &= \{(x, y, z) : -1 \leq x \leq 1, 0 \leq y \leq x^2, 0 \leq z \leq y\}. \end{aligned}$$

$$D_x = \{(y, z) : 0 \leq y \leq x^2, 0 \leq z \leq y\}, \quad -1 \leq x \leq 1.$$



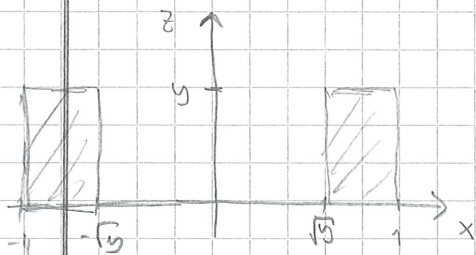
b)

$$0 \leq y \leq 1,$$

$$D_y = \{(x, z) : 0 \leq z \leq y, y \leq x^2 \leq 1\} =$$

$$= \{(x, z) : 0 \leq z \leq y, \sqrt{y} \leq x \leq 1\}$$

$$\cup \{(x, z) : 0 \leq z \leq y, \sqrt{y} \leq -x \leq 1\}$$

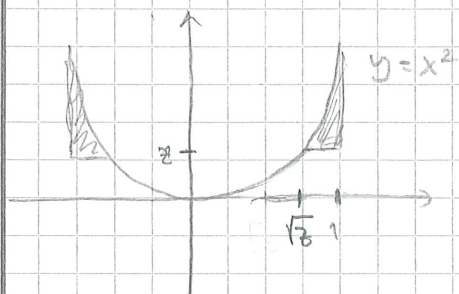


c)

$$0 \leq z \leq 1,$$

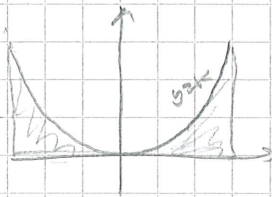
$$D_z = \{(x, y) : z \leq y \leq x^2, z \leq x^2 \leq 1\}$$

$$= \{(x, y) : z \leq y \leq x^2, \sqrt{z} \leq x \leq 1\} \cup \{(x, y) : z \leq y \leq x^2, \sqrt{z} \leq -x \leq 1\}$$



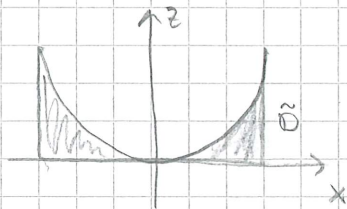
11.5)

d) $\tilde{D} = \{(x, y) : 0 \leq y \leq x^2, 0 \leq x^2 \leq 1\} =$
 $= \{(x, y) : 0 \leq y \leq x^2, -1 \leq x \leq 1\}$



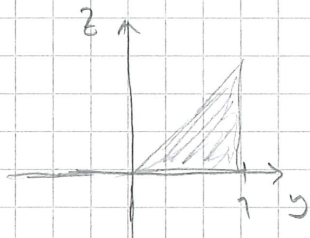
$D_{xy} = \{z : 0 \leq z \leq y\}$

e) $\tilde{D} = \{(x, z) : 0 \leq z \leq x^2 \leq 1\}$



$D_{xz} = \{y : z \leq y \leq x^2\}$

f) $\tilde{D}_{yz} = \{(y, z) : 0 \leq z \leq y \leq 1\}$, $D_{yz} = \{x : -1 \leq x \leq -\sqrt{y}\} \cup$
 $\cup \{x : \sqrt{y} \leq x \leq 1\}$



11.6)

$$\iiint_D \frac{z}{1+x^2} dx dy dz =$$

$$D = \{(x, y, z) : 0 \leq z \leq y \leq x^2 \leq 1\}$$

$$\tilde{D} = \{(x, y) : 0 \leq y \leq x^2, -1 \leq x \leq 1\}, \quad 0 \leq z \leq y$$

$$= \int_{\tilde{D}} \left(\int_0^y \frac{z}{1+x^2} dz \right) dx dy = \int_{\tilde{D}} \frac{y^2}{2(1+x^2)} dx dy =$$

$$= \int_{-1}^1 \left(\int_0^{x^2} \frac{y^2}{2(1+x^2)} dy \right) dx = \int_{-1}^1 \frac{x^6}{6(1+x^2)} dx$$

$$= \frac{x^6}{6(1+x^2)} = \frac{x^6(1+x^2) - x^4}{6(1+x^2)} = \frac{x^4}{6} - \frac{x^2(1+x^2) - x^2}{6(1+x^2)} =$$

$$= \frac{x^4 - x^2}{6} + \frac{x^2}{6(1+x^2)} = \frac{x^4 - x^2 + 1}{6} - \frac{1}{6(1+x^2)}$$

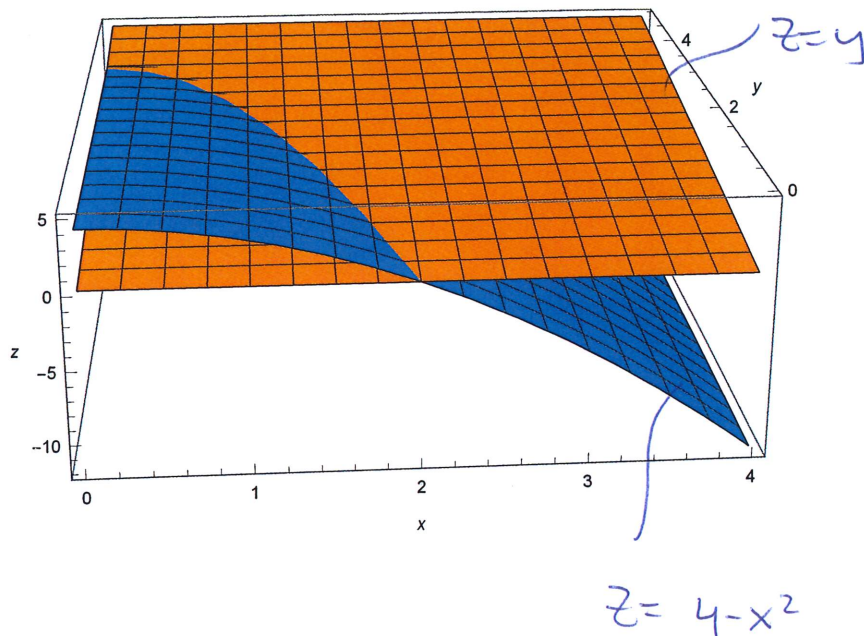
$$= \left[\frac{x^5}{30} - \frac{x^3}{18} + \frac{x}{6} - \frac{1}{6} \arctan x \right]_{-1}^1 =$$

$$2 \left(\frac{1}{30} - \frac{1}{18} + \frac{1}{6} - \frac{\pi}{24} \right) = \frac{1}{15} - \frac{1}{9} + \frac{1}{3} - \frac{\pi}{12} = \frac{3-5+15}{45} - \frac{\pi}{12} =$$

$$= \frac{13}{45} - \frac{\pi}{12}$$

~~11.7~~ (Alternativ lösning).

11.7



Projektion $\tilde{D}$ på xy -planet ges av $0 \leq x \leq 2$, $0 \leq y \leq 4-x^2$.

$$y \leq z \leq 4-x^2.$$

$$\iint_{\tilde{D}} \left(\int_y^{4-x^2} 2x \, dz \right) dx dy = \int_0^2 \left(\int_0^{4-x^2} [2xz]_{z=y}^{4-x^2} dy \right) dx =$$

$$= \int_0^2 \left(\int_0^{4-x^2} (2x(4-x^2) - 2xy) dy \right) dx = \int_0^2 [2x(4-x^2)y - xy^2]_{y=0}^{4-x^2} dx$$

$$= \int_0^2 (2x(4-x^2)^2 - x(4-x^2)^2) dx = \dots = \underline{\underline{\frac{32}{3}}}$$

11.7)

D begränsas av $x \geq 0, y \geq 0, z \geq 0, z = y$ och $z = 4 - x^2$

Projektion $\tilde{D}$ på xz -planet ges av

$0 \leq x \leq 2, 0 \leq z \leq 4 - x^2$, och för varje

$(x, z) \in \tilde{D}$ får vi gränserna $0 \leq y \leq z$.

$$\iiint_D 2x \, dx \, dy \, dz = \iint_{\tilde{D}} \left(\int_0^z 2x \, dy \right) dx \, dz =$$

$$= \int_0^2 \left(\int_0^{4-x^2} 2xz \, dz \right) dx = \int_0^2 \left[xz^2 \right]_{z=0}^{4-x^2} dx = \int_0^2 x(4-x^2)^2 dx =$$

$$= \left[\frac{-(4-x^2)^3}{3 \cdot 2} \right]_0^2 = \frac{32}{3}$$

11.8)

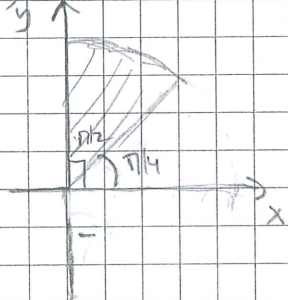
a)

$$\begin{aligned} \iiint_{B(0,1)} (z^2 - x^2 - y^2) dx dy dz &= \int_0^\pi \left(\int_0^{2\pi} \left(\int_0^1 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) r^2 \sin \theta dr \right) d\varphi \right) d\theta \\ &= 2\pi \int_0^\pi (2\cos^2 \theta - 1) \sin \theta d\theta \cdot \int_0^1 r^4 dr = \\ &= 2\pi \left[-\cos \theta - \frac{2\cos^3 \theta}{3} \right]_0^\pi \cdot \left[\frac{r^5}{5} \right]_0^1 = \underline{\underline{-\frac{4\pi}{15}}} \end{aligned}$$

b) Gränser i sfäriska koordinater $0 \leq r \leq 1$, $-\pi/2 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq \pi$

$$\begin{aligned} &\int_0^\pi \left(\int_{-\pi/2}^{\pi/2} \left(\int_0^1 r \cos \varphi \sin \theta e^{r^2} \cdot r^2 \sin \theta dr \right) d\varphi \right) d\theta = \\ &= \int_0^\pi \sin^2 \theta d\theta \cdot \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \cdot \int_0^1 r^3 e^{r^2} dr = \\ &= \int_0^\pi \frac{1 - \cos 2\theta}{2} d\theta \cdot \left[\sin \varphi \right]_{-\pi/2}^{\pi/2} \cdot \left(\left[r^2 \frac{e^{r^2}}{2} \right]_0^1 - \int_0^1 r e^{r^2} dr \right) = \\ &= \frac{\pi}{2} \cdot 2 \cdot \left[r^2 \frac{e^{r^2}}{2} - \frac{e^{r^2}}{2} \right]_0^1 = \pi \left(\frac{e}{2} - \frac{e}{2} - 0 + \frac{1}{2} \right) = \underline{\underline{\frac{\pi}{2}}} \end{aligned}$$

c)



$0 \leq r \leq \sqrt{3}$, $-\pi/4 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq \pi$.

$$\int_0^\pi \left(\int_{\pi/4}^{\pi/2} \left(\int_0^{\sqrt{3}} r \cos \varphi \sin \theta r^2 \sin \theta dr \right) d\varphi \right) d\theta =$$

$$\begin{aligned} &= \int_0^\pi \sin^2 \theta d\theta \int_{\pi/4}^{\pi/2} \cos \varphi d\varphi \int_0^{\sqrt{3}} r^3 dr = \frac{\pi}{2} \cdot \left[\sin \varphi \right]_{\pi/4}^{\pi/2} \cdot \left[\frac{r^4}{4} \right]_0^{\sqrt{3}} = \\ &= \frac{\pi}{2} \cdot \left(1 - \frac{1}{\sqrt{2}} \right) \cdot \frac{9}{4} = \underline{\underline{\frac{9\pi(2-\sqrt{2})}{16}}} \end{aligned}$$

1.9)

$$\begin{aligned} \text{a)} \quad \begin{cases} u = x + y + z \\ v = x + 2y + 3z \\ w = x + 4y + 9z \end{cases} &\Leftrightarrow \dots \Leftrightarrow \begin{cases} x = 3u - \frac{5v}{2} + \frac{w}{2} \\ y = -3u + 4v - w \\ z = \frac{2u - 3v + w}{2} \end{cases} \end{aligned}$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

Så $dx dy dz = \frac{1}{2} du dv dw$

$0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1$

$y - x - z = \dots = -7u + 8v - 2w$

$$\iiint_0^1 (y-x-z) dx dy dz = \int_0^1 \int_0^1 \int_0^1 (-7u + 8v - 2w) \frac{1}{2} du dv dw$$

= / Av symmetriskhet / = $-\frac{1}{2} \int_0^1 u du = -\frac{1}{4}$

b)

Med basbytet till basen

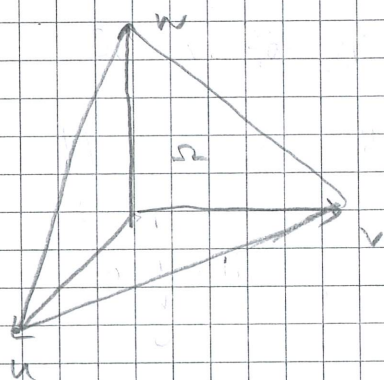
$((1,1,0), (1,0,1), (0,1,1))$ får vi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \Leftrightarrow \begin{cases} x = u + v \\ y = u + w \\ z = v + w \end{cases}$$

$$\frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

Punkterna $(0,0,0), (1,1,0), (1,0,1)$ och $(0,1,1)$

är i uvw koordinaterna $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.



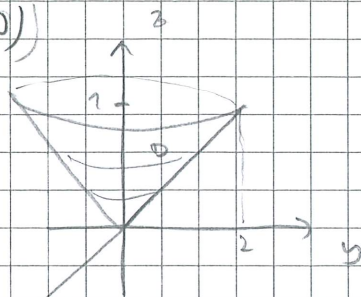
$\Omega: 0 \leq u \leq 1, 0 \leq v \leq 1-u, 0 \leq w \leq 1-u-v$

$$\iiint_{\Omega} x dx dy dz = \iiint_{\Omega} (u+v) 2 du dv dw$$

= Symmetri / = $\frac{1}{2} \iiint_{\Omega} u du dv dw =$

$$= 4 \int_0^1 \left(\int_0^{1-u} \left(\int_0^{1-u-v} u dv \right) dw \right) du = 4 \int_0^1 \left(\int_0^{1-u} u(1-u-v) dv \right) du = \dots = \frac{1}{6}$$

16.10)



$$D: 0 \leq z \leq 1, \sqrt{x^2 + y^2} \leq 2z$$

$$\int_0^1 \left(\int_0^{2\pi} \left(\int_0^{2z} (p^2 \sin^2 \phi + z^2) p dp \right) d\phi \right) dz =$$

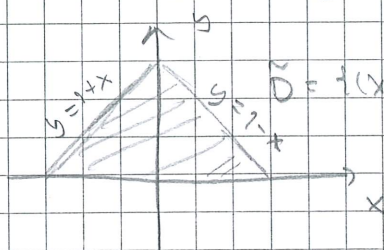
$$= \int_0^1 \left(\int_0^{2\pi} \left[\frac{p^4}{4} \sin^2 \phi + \frac{p^2}{2} z^2 \right]_{p=0}^{2z} d\phi \right) dz = \int_0^1 \left(\int_0^{2\pi} (4z^4 \sin^2 \phi + 2z^4) d\phi \right) dz$$

$$= \int_0^1 2z^4 dz \cdot \int_0^{2\pi} (1 + \sin^2 \phi) d\phi = \left[\frac{2z^5}{5} \right]_0^1 \cdot \left[\phi + \phi^2 - \frac{\sin 2\phi}{2} \right]_0^{2\pi}$$

$$= \frac{8\pi}{5}$$

$$(1.11) \quad x-y \leq z \leq x+y, \quad |x|+|y| \leq 1.$$

$$x-y \leq x+y \Leftrightarrow y \geq 0$$



$$\tilde{D} = \{(x,y) : y \geq 0, |x|+|y| \leq 1\} =$$

$$= \{(x,y) : 0 \leq y \leq 1, y-1 \leq x \leq 1-y\}$$

$$\iiint_D y e^z dx dy dz = \iint_{\tilde{D}} \left(\int_{x-y}^{x+y} y e^z dz \right) dx dy =$$

$$= \iint_{\tilde{D}} y (e^{x+y} - e^{x-y}) dx dy = \int_0^1 \left(\int_{y-1}^{1-y} y (e^x - e^{-x}) dx \right) dy =$$

$$= \int_0^1 y (e^y - e^{-y}) \cdot (e^{1-y} - e^{y-1}) dy = \dots = \frac{1}{e}$$