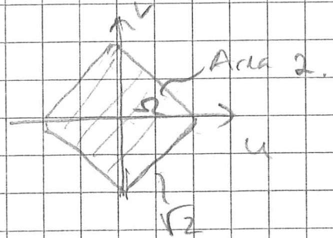
12.1)

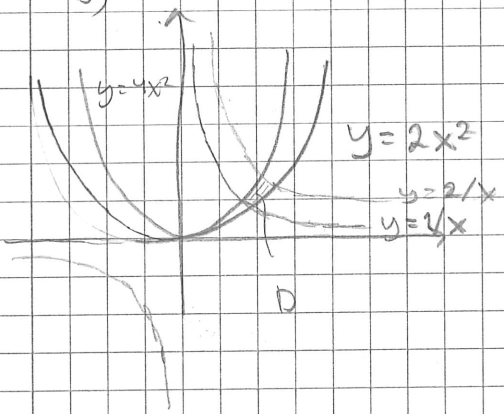
a) $\begin{cases} u = x + 2y \\ v = 3x - y \end{cases}$ överför området på $|u| + |v| \leq 1$.

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7$$



$$\iint_D dxdy = \iint_\Omega \frac{1}{7}\, dudv = \underline{\underline{\frac{2}{7}}}$$

b)



$\begin{cases} u = y/x^2 \\ v = xy \end{cases}$

$y = 2x^2 \Leftrightarrow y/x^2 = u = 2$

$y = 4x^2 \Leftrightarrow y/x^2 = u = 4$

$xy = 1 \Leftrightarrow v = 1, \quad xy = 2 \Leftrightarrow v = 2$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ y & x \end{vmatrix} = \frac{-2y}{x^2} - \frac{y}{x^2} = \frac{-3y}{x^2} = 3u$$

D.v.s $dxdy = \frac{1}{3u} dudv$.

$$\iint_D dxdy = \int_1^2 \left(\int_2^4 \frac{1}{3u} du \right) dv = \underline{\ln 2/3}$$

12.2)

a)

$x = au,\ y = bv,\ z = cw$ överför

D på $u^2+v^2+w^2 \le 1$, som har volym $\frac{4\pi}{3}$ i uvw-rummet

$$\frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Så Volym$(D) = \frac{4\pi abc}{3}$.

b)

$$3\left(x+\frac{z}{3}\right)^2 - \frac{z^2}{3} + 2y^2 + 2z^2 - 2yz =$$

$$= 3\left(x+\frac{z}{3}\right)^2 + 2\left(y-\frac{z}{2}\right)^2 - \frac{z^2}{2} - \frac{z^2}{3} + 2z^2 =$$

$$= 3\left(x+\frac{z}{3}\right)^2 + 2\left(y-\frac{z}{2}\right)^2 + \frac{7z^2}{6} \le 1$$

$$\begin{cases} u = \left(x+\frac{z}{3}\right) \\ v = y - \frac{z}{2} \\ w = z \end{cases} \qquad 3u^2 + 2v^2 + \frac{7w^2}{6} \le 1$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Enligt (a) har $3u^2+2v^2+\frac{7w^2}{6} \le 1$ volym

$\frac{4\pi abc}{3}$ då $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{2}}$, $c = \frac{\sqrt{6}}{\sqrt{7}}$

D.v.s. $\dfrac{4\pi \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \sqrt{6}}{3\sqrt{7}} = \underline{\underline{\dfrac{4\pi}{3\sqrt{7}}}}.$

12.3)

$$\begin{cases} u = 2x+y+z \\ v = x+2y+z \\ w = x+y+2z \end{cases}$$

$u+v+w = 4x+4y+4z$.

Så området avbildas på området Ω i uvw-rummet

som avgränsas av planen $u=0$, $v=0$, $w=0$ och $u+v+w=16$.

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 4 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4.$$

Den sökta volymen ges alltså av

$$\frac{1}{4}\iiint_\Omega du\,dv\,dw = \frac{1}{4}\int_0^{16}\left(\int_0^{16-u}\left(\int_0^{16-u-v} dw\right)dv\right)du =$$

$$= \frac{1}{4}\int_0^{16}\left(\int_0^{16-u}(16-u-v)\,dv\right)du = \frac{1}{4}\int_0^{16}\left[(16-u)v - \frac{v^2}{2}\right]_0^{16-u} du$$

$$= \frac{1}{4}\int_0^{16}\frac{(16-u)^2}{2}\,du = \frac{1}{4}\left[-\frac{(16-u)^3}{6}\right]_0^{16} = \frac{1}{4}\,\frac{16^3}{6} = \frac{2\cdot 16^2}{3} = \frac{512}{3}$$

(12.4)
a)

$9x^2+4y^2 \le 36$: Ω, projektion på xy-planet

$z = 10-x-y$, $z = x+y-10$

Eftersom $x^2 \le \frac{36}{9} = 4 \iff -2 \le x \le 2$

$y^2 \le \frac{36}{4} = 9 \iff -3 \le y \le 3$

ser vi att det första av dessa plan ligger

över det andra, d.v.s. $x+y-10 \le z \le 10-x-y$.

Volymen ges av

$$\iint_\Omega \left(\int_{x+y-10}^{10-x-y} dz \right) dx\,dy = \iint_\Omega (20-2x-2y)\,dx\,dy =$$

$$= \left/ \begin{array}{l} x = \frac{\rho}{3}\cos\varphi \\ y = \frac{\rho}{2}\sin\varphi \end{array} \quad \begin{array}{l} \Omega \text{ ges av} \\ \rho^2 \le 36, \text{ dvs } 0 \le \rho \le 6,\ 0 \le \varphi \le 2\pi. \end{array} \quad \frac{d(x,y)}{d(\rho,\varphi)} = \frac{1}{6}\rho \right/$$

$$= \int_0^{2\pi} \left(\int_0^6 \left(20 - 2\frac{\rho}{3}\cos\varphi - 2\frac{\rho}{2}\sin\varphi\right)\frac{1}{6}\rho\,d\rho \right) d\varphi =$$

$$= \frac{1}{6}\int_0^{2\pi} \left[10\rho^2 - \frac{2\rho^3}{9}\cos\varphi - \frac{2\rho^3}{6}\sin\varphi\right]_{\rho=0}^{6} d\varphi =$$

$$= \frac{1}{6}\int_0^{2\pi} \left(10\cdot 6^2 - \frac{2\cdot 6^3}{9}\cos\varphi - \frac{2\cdot 6^3}{6}\sin\varphi\right) d\varphi =$$

$$= \left[60\varphi - 8\sin\varphi + 12\cos\varphi\right]_0^{2\pi} = \underline{\underline{120\pi}}$$

(12.4)

b)

$$x^2+y^2=1-2x-2y \Leftrightarrow (x+1)^2+(y+1)^2=3$$

Området har alltså projektion som är cirkeln

med centrum i (-1,-1) och radie $\sqrt{3}$.

$0 \leq z \leq 1-2x-2y$ för varje (x,y) i denna projektion

Volymen ges av

$$V=\iint_{\Omega}\left(\int_{x^2+y^2}^{1-2x-2y} dz\right)dxdy$$ där Ω är projektionen på

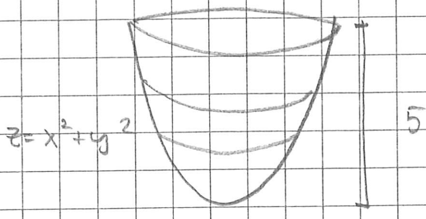
xy-planet.

$$V=\iint_{\Omega}(1-2x-2y-x^2-y^2)\,dxdy=\iint_{\Omega}(3-(x+1)^2-(y+1)^2)$$

$$=\left/\begin{matrix} x+1=\rho\cos\varphi \\ y+1=\rho\sin\varphi \end{matrix}\right/=\int_0^{2\pi}\left(\int_0^{\sqrt{3}}(3-\rho^2)\rho\,d\rho\right)d\varphi=$$

$$=2\pi\left[\frac{3\rho^2}{2}-\frac{\rho^4}{4}\right]_0^{\sqrt{3}}=2\pi\left(\frac{9}{2}-\frac{9}{4}\right)=\underline{\underline{\frac{9\pi}{2}}}!$$

12.5)



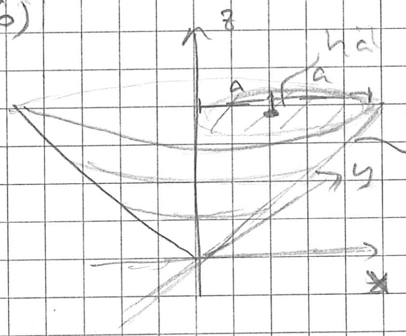
$D = \{(x,y,z) : 0 \le z \le 5,\ x^2+y^2 \le z\}$.

$D_z = \{(x,y) : x^2+y^2 \le z\}$

D_z har area $\pi \cdot \sqrt{z}^2 = \pi z$

$$\iiint_D dx\,dy\,dz = \int_0^5 \left(\iint_{D_z} dx\,dy \right) dz = \int_0^5 \pi z\,dz = \underline{\underline{\frac{25\pi}{2}}}.$$

12.6)



$z = \frac{h}{2a}\sqrt{x^2+y^2}$.

Konen ges av olikheterna $x^2+y^2 \le (2a)^2 = 4a^2$, $\frac{h}{2a}\sqrt{x^2+y^2} \le z \le h$.

Så den har volym

$$\iint_{B(0,2a)} \left(\int_{\frac{h\sqrt{x^2+y^2}}{2a}}^{h} dz \right) dx\,dy = \int_0^{2\pi} \left(\int_0^{2a} \left(h - \frac{h\rho}{2a}\right) \rho\,d\rho \right) d\varphi = \ldots = \frac{4\pi h a^2}{3}.$$

Hålet ges av olikheterna $(x-a)^2+y^2 \le a^2$ $\frac{h}{2a}\sqrt{x^2+y^2} \le z \le h$.

I polära koordinater i xy-planet ges området:

$(x-a)^2+y^2 \le a^2 \iff (\rho\cos\varphi - a)^2 + \rho^2\sin^2\varphi = \rho^2 - 2a\rho\cos\varphi + a^2 \le a^2$

$\iff 0 \le \rho \le 2a\cos\varphi,\quad -\pi/2 \le \varphi \le \pi/2$.

Volymen av hålet blir nu

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^{2a\cos\varphi} \left(\int_{\frac{h\rho}{2a}}^{h} dz \right) \rho\,d\rho \right) d\varphi = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2a\cos\varphi} \left(h - \frac{h\rho}{2a}\right) \rho\,d\rho \right) d\varphi$$

$$= \ldots = \left(\pi - \frac{16}{9}\right) h a^2.$$

Så volym kon − volym hål $= \frac{4\pi h a^2}{3} - \left(\pi - \frac{16}{9}\right) h a^2 = \underline{\underline{\left(\frac{\pi}{3} + \frac{16}{9}\right) a^2 h}}$

12.7)

a) Eftersom $x, y, z \geq 0$, gäller att

$x + y^2 + z = 1$ tvingar $x \leq 1, y^2 \leq 1, z \leq 1 \Leftrightarrow x \leq 1, y \leq 1, z \leq 1$.

$0 \leq V \leq 1$.

b) Projektion på xz-planet $x \geq 0, z \geq 0, x + z \leq 1$

$0 \leq y^2 \leq 1 - x - z \Leftrightarrow$ (eftersom $y \geq 0$) $\Leftrightarrow 0 \leq y \leq \sqrt{1-x-z}$

$$\iiint_D dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{\sqrt{1-x-z}} dy \right) dz \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \sqrt{1-x-z}\, dz \right) dx = \int_0^1 \left[-\frac{2}{3}(1-x-z)^{3/2} \right]_{z=0}^{1-x} dx$$

$$= \int_0^1 \frac{2}{3}(1-x)^{3/2} dx = \left[-\frac{2}{3} \cdot \frac{2}{5} (1-x)^{5/2} \right]_0^1 = \underline{\underline{\frac{4}{15}}}$$

12.8)

$D = \{(x,y,z) : z \geq y^2, y \geq x^2, x \geq z^2\} =$

$= \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, z^2 \leq x \leq \sqrt{y} \leq \sqrt[4]{z}\}$

Projektion på xz-planet $\tilde{D}$ ges av $0 \leq z \leq 1$, $z^2 \leq x \leq \sqrt[4]{z}$,

om för varje $(x,z) \in \tilde{D}$ uppfyller y

$x^2 \leq y \leq \sqrt{z}$.

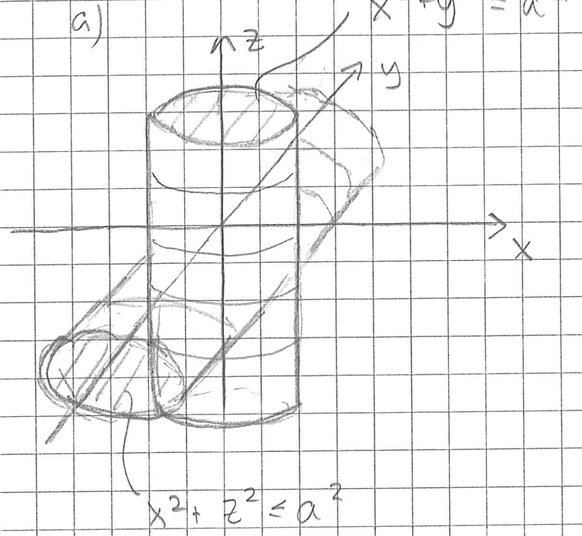
$$\int_0^1 \left(\int_{z^2}^{\sqrt[4]{z}} \left(\int_{x^2}^{\sqrt{z}} dy \right) dx \right) dz = \int_0^1 \left(\int_{z^2}^{\sqrt[4]{z}} (\sqrt{z} - x^2)\, dx \right) dz$$

$$= \int_0^1 \left[\sqrt{z}\,x - \frac{x^3}{3} \right]_{x=z^2}^{\sqrt[4]{z}} dz = \int_0^1 \left(\frac{2z^{3/4}}{3} - z^{5/2} + \frac{z^6}{3} \right) dz =$$

$$= \left[\frac{2}{3} \cdot \frac{4}{7} \cdot z^{7/4} - \frac{2}{7} \cdot z^{7/2} + \frac{z^7}{21} \right]_0^1 = \frac{8}{21} - \frac{2}{7} + \frac{1}{21} = \frac{3}{21} = \underline{\underline{\frac{1}{7}}}$$

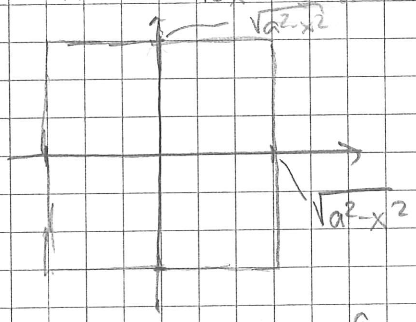
12.29)

a)



Projektion på x-axeln $-a \le x \le a$.

Snitt $D_x = \{(y,z) : -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},\ -\sqrt{a^2-x^2} \le z \le \sqrt{a^2-x^2}\}$.



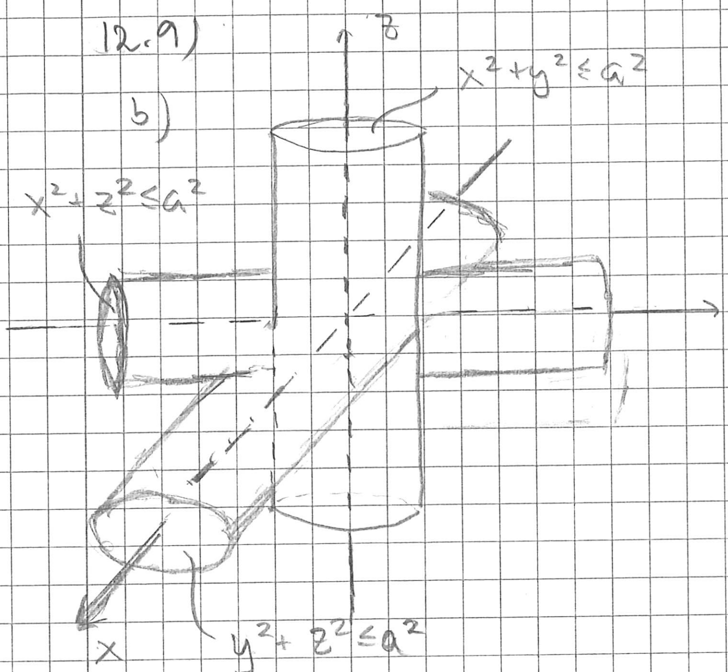
D_x är kvadrat med sidlängd $2\sqrt{a^2-x^2}$. Area $4(a^2-x^2)$.

$$\iiint_D dx\,dy\,dz = \int_{-a}^{a}\left(\iint_{D_x} dy\,dz\right)dx = \int_{-a}^{a} 4(a^2-x^2)\,dx =$$

$$= 4\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \underline{\underline{\frac{16a^3}{3}}}$$

12.9)

b)



$D = \{(x,y,z):\ x^2+y^2\le a^2,\ x^2+z^2\le a^2,\ y^2+z^2\le a^2\}$

Projektion $\tilde{D} = \{(x,y):\ x^2+y^2\le a^2\}$

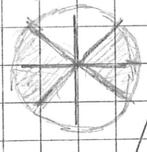
Gränserna för z ges av att både

$x^2+z^2\le a^2$ och $y^2+z^2\le a^2$. D.v.s.

$-\min(\sqrt{a^2-x^2},\sqrt{a^2-y^2}) \le z \le \min(\sqrt{a^2-x^2},\sqrt{a^2-y^2})$.

$$\iiint_D dx\,dy\,dz = \iint_{\tilde{D}}\left(\int_{-\min(\sqrt{a^2-x^2},\sqrt{a^2-y^2})}^{\min(\sqrt{a^2-x^2},\sqrt{a^2-y^2})} dz\right)dx\,dy =$$

$$= 2\iint_{\tilde{D}} \min(\sqrt{a^2-x^2},\sqrt{a^2-y^2})\,dx\,dy = \Bigg/\ a^2-x^2\le a^2-y^2 \Leftrightarrow y^2\le x^2 \Leftrightarrow -\pi/4\le\varphi\le\pi/4 \text{ eller } \frac{3\pi}{4}\le\varphi\le\frac{5\pi}{4}.\ \Bigg/$$



$$= /\text{Av symmetriskäl}/ = 16\int_0^{\pi/4}\left(\int_0^a \sqrt{a^2-\rho^2\cos^2\varphi}\ \rho\,d\rho\right)d\varphi =$$

$$= 16\int_0^{\pi/4}\left[-(a^2-\rho^2\cos^2\varphi)^{3/2}\cdot\frac{1}{3\cos^2\varphi}\right]_{\rho=0}^{a} d\varphi =$$

$$= 16\int_0^{\pi/4}\left(\frac{a^3}{3\cos^2\varphi} - a^3\cdot\frac{(1-\cos^2\varphi)^{3/2}}{3\cos^2\varphi}\right)d\varphi = \ldots =$$

12.9)

b) forts.

$$= \frac{16a^3}{3}\int_0^{\pi/4} \frac{1}{\cos^2\varphi}\,d\varphi - \frac{16a^3}{3}\int_0^{\pi/4} \frac{\sin^3\varphi}{\cos^2\varphi}\,d\varphi =$$

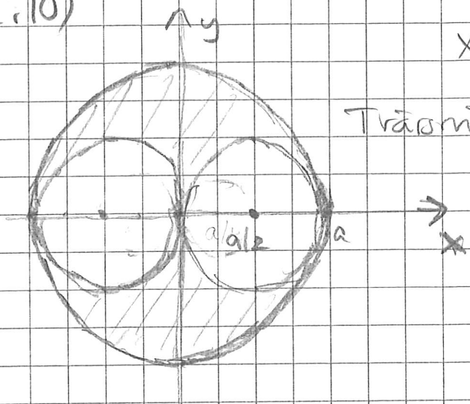
$$= \frac{16a^3}{3}\Big[\tan\varphi\Big]_0^{\pi/4} - \frac{16a^3}{3}\int_0^{\pi/4} \frac{(1-\cos^2\varphi)\sin\varphi}{\cos^2\varphi}\,d\varphi =$$

$$= \frac{16a^3}{3} - \frac{16a^3}{3}\int_0^{\pi/4}\left(\frac{\sin\varphi}{\cos^2\varphi} - \sin\varphi\right)d\varphi =$$

$$= \frac{16a^3}{3} - \frac{16a^3}{3}\left[\frac{1}{\cos\varphi} + \cos\varphi\right]_0^{\pi/4} =$$

$$= \frac{16a^3}{3}\left(1 - \frac{1}{\sqrt{2}} - \sqrt{2} + 1 + 1\right) = \underline{\underline{8a^3(2-\sqrt{2})}}$$

12.10)



$x^2+y^2+z^2 \le a^2$

Tvärsnitt för $z=0$

Hål $(x-\frac{a}{2})^2+y^2 \le \frac{a^2}{4}$

$(x+\frac{a}{2})^2+y^2 \le \frac{a^2}{4}$

I detta fall är tvärsnittet för $z=0$ också projektionen av D på xy-planet.

Volymen ges nu av Volym(Klot) − Volym hål:

$$\frac{4\pi a^3}{3} - 2\iint_B \left(\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\right) dx\,dy \quad / \text{där } B: (x-\tfrac{a}{2})^2+y^2 \le \tfrac{a^2}{4} /$$

$$= \frac{4\pi a^3}{3} - 4\iint_B \sqrt{a^2-x^2-y^2}\,dx\,dy = \left/ \begin{array}{l} (\rho\cos\varphi-\frac{a}{2})^2+\rho^2\sin^2\varphi \le \frac{a^2}{4} \\ \Leftrightarrow \rho^2 \le a\rho\cos\varphi \Leftrightarrow \\ 0\le\rho\le a\cos\varphi,\ -\pi/2\le\varphi\le\pi/2 \end{array} \right/$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2}\left(\int_0^{a\cos\varphi}\sqrt{a^2-\rho^2}\,\rho\,d\rho\right)d\varphi =$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2}\left[-\frac{1}{3}(a^2-\rho^2)^{3/2}\right]_{\rho=0}^{a\cos\varphi} d\varphi =$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2}\left(\frac{a^3}{3}-\frac{a^3}{3}(1-\cos^2\varphi)^{3/2}\right)d\varphi =$$

$$= \frac{4\pi a^3}{3} - \frac{4\pi a^3}{3} + \frac{4a^3}{3}\int_{-\pi/2}^{\pi/2}(1-\cos^2\varphi)^{3/2}d\varphi =$$

$$= \frac{8a^3}{3}\int_0^{\pi/2}(1-\cos^2\varphi)\sin\varphi\,d\varphi = \frac{8a^3}{3}\left[-\cos\varphi+\frac{\cos^3\varphi}{3}\right]_0^{\pi/2} = \underline{\underline{\frac{16a^3}{9}}}$$