4.1 a) $f(x,y) = x + x^3y + x^2y^3 + y^5$

$f'_x = 1 + 3x^2y + 2xy^3$

$f'_y = x^3 + 3x^2y^2 + 5y^4$

b) $f(x,y) = \ln(1 - x^2 - 2y^2)$

$f'_x = \dfrac{1}{1-x^2-2y^2} \cdot (-2x) = \dfrac{-2x}{1-x^2-2y^2}$

$f'_y = \dfrac{1}{1-x^2-2y^2} (-4y) = \dfrac{-4y}{1-x^2-2y^2}$

c) $f(x,y) = \dfrac{x+y}{x-y}$

$f'_x = \dfrac{1\cdot(x-y) - 1\cdot(x+y)}{(x-y)^2} = \dfrac{-2y}{(x-y)^2}$

$f'_y = \dfrac{1\cdot(x-y) + 1(x+y)}{(x-y)^2} = \dfrac{2x}{(x-y)^2}$

4.2 a)

$f(x,y,z) = x^2 y^3 z^4$

$f'_x = 2x y^3 z^4$

$f'_y = 3x^2 y^2 z^4$

$f'_z = 4x^2 y^3 z^3$

4.2 b)

$$f(x,y,z) = \cos(xy - z^2)$$

$$f'_x = -\sin(xy - z^2) \cdot y = -y\sin(xy - z^2)$$

$$f'_y = -x\sin(xy - z^2)$$

$$f'_z = (-2z) \cdot (-\sin(xy - z^2)) = 2z\sin(xy - z^2)$$

c)

$$f(x,y,z) = \frac{1}{\sqrt{z}} \arctan\frac{y}{x}$$

$$f'_x = \frac{1}{\sqrt{z}} \cdot \frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{\sqrt{z}(x^2+y^2)}$$

$$f'_y = \frac{1}{\sqrt{z}} \cdot \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{\sqrt{z}(x^2+y^2)}$$

$$f'_z = -\frac{1}{2z\sqrt{z}} \arctan\frac{y}{x}.$$

4.3

$$f''_{xx} = (f'_x)'_x = \frac{\partial}{\partial x}(1 + 3x^2y + 2xy^3 + 5y^4) = 6xy + 2y^3$$

$$f''_{xy} = (f'_x)'_y = \frac{\partial}{\partial y}(1 + 3x^2y + 2xy^3) = 3x^2 + 6xy^2$$

$$f''_{yx} = (f'_y)'_x = \frac{\partial}{\partial x}(x^3 + 3x^2y^2 + 5y^4) = 3x^2 + 6xy^2$$

$$f''_{yy} = (f'_y)'_y = \frac{\partial}{\partial y}(x^3 + 3x^2y^2 + 5y^4) = 6x^2y + 20y^3.$$

4.4)

a) $\begin{cases} z'_x = 2x + y \\ z'_y = x + 2y \end{cases}$

$z'_x = 2x + y \Rightarrow z = x^2 + xy + g(y) \Rightarrow$

$\Rightarrow z'_y = x + g'(y) = x + 2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C$

SVAR: $z = x^2 + xy + y^2 + C$.

b) $\begin{cases} z'_x = e^{xy} \\ z'_y = e^{xy} \end{cases}$

$z'_x = e^{xy} \Rightarrow z = \frac{1}{y} e^{xy} + g(y) \Rightarrow z'_y = -\frac{1}{y^2} e^{xy} + \frac{x}{y} e^{xy} + g'(y)$

$= e^{xy} \Leftrightarrow g'(y) = e^{xy}\left(1 + \frac{1}{y^2} - \frac{x}{y}\right)$

Går ej, så lösning saknas.

c) $\begin{cases} z'_x = ye^x \\ z'_y = 1 + e^x \end{cases}$

$z'_x = ye^x \Rightarrow z = ye^x + g(y) \Rightarrow z'_y = e^x + g'(y) = 1 + e^x$

$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y + C$.

SVAR: $z = ye^x + y + C$.

4.5)

a) $\begin{cases} u'_x = y+3z-3 \\ u'_y = x+2z-2 \\ u'_z = 2y+3x-1 \end{cases}$

$u'_x = y+3z-3 \Rightarrow u = xy+3xz-3x+h(y,z) \Rightarrow$

$\Rightarrow u'_y = x + h'_y = x+2z-2 \Rightarrow h'_y = 2z-2$

$\Rightarrow h = 2yz-2y+g(z) \Rightarrow u = xy+3xz-3x+2yz-2y+g(z)$

$\Rightarrow u'_z = 3x+2y+g'(z) = 2y+3x-1$

$\Rightarrow g'(z) = -1 \Rightarrow g(z) = -z+C$

SVAR: $u = xy+3xz-3x+2yz-2y-z+C$

b) $\begin{cases} u'_x = 1+y\sin xy \\ u'_y = e^z + x\sin xy \\ u'_z = (1+y+z)e^z \end{cases}$.

$u'_x = 1+y\sin xy \Rightarrow u = x - \cos xy + h(y,z)$

$\Rightarrow u'_y = x\sin xy + h'_y = e^z + x\sin xy \Rightarrow h'_y = e^z$

$\Rightarrow h = ye^z + g(z) \Rightarrow u = x-\cos xy + ye^z + g(z)$

$\Rightarrow u'_z = ye^z + g'(z) = (1+y+z)e^z$

$\Rightarrow g'(z) = (1+z)e^z \Rightarrow g(z) = ze^z + C$

SVAR: $u = x - \cos xy + ye^z + ze^z + C$.

c) $\begin{cases} u'_x = z+xy^2 \\ u'_y = x^2y \\ u'_z = yz \end{cases}$

$u'_x = z+xy^2 \Rightarrow u = xz + \frac{x^2y^2}{2} + h(y,z)$

$\Rightarrow u'_y = x^2y + h'_y = x^2y \Rightarrow h'_y = 0$

$\Rightarrow h = g(z) \Rightarrow u = xz + \frac{x^2y^2}{2} + g(z)$

$\Rightarrow u'_z = x + g'(z) = yz$

Går ej, så lösning saknas.

4.6 a) $z = z(x,y)$

$$\frac{\partial z}{\partial x} = 0 \iff z = h(y) \quad h \in C^2(\mathbb{R})$$

b) $$\frac{\partial z}{\partial y} = 0 \iff z = h(x) \quad h \in C^2(\mathbb{R})$$

c) $$\frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow \frac{\partial z}{\partial x} = h(y) \Rightarrow z = x\,h(y) + g(y) \quad h, g \in C^2(\mathbb{R})$$

d) $$\frac{\partial^2 z}{\partial x \partial y} = 0 \Rightarrow \frac{\partial z}{\partial x} = h(x) \Rightarrow z = H(x) + K(y) \quad H, K \in C^2(\mathbb{R})$$

e) $$\frac{\partial z}{\partial x} = z \iff e^{-x}\frac{\partial z}{\partial x} = e^{-x} z \iff \frac{\partial}{\partial x}(e^{-x} z) = 0$$

$$\iff e^{-x} z = h(y) \qquad \underline{\underline{z = h(y)e^{x} \quad h \in C^2(\mathbb{R})}}$$

f) $$\frac{\partial z}{\partial x} = yz \iff e^{-xy}\frac{\partial z}{\partial x} = e^{-xy} yz$$

$$\iff \frac{\partial}{\partial x}(e^{-xy} z) = 0 \iff e^{-xy} z = h(y)$$

$$\underline{\underline{z = h(y)e^{xy} \quad h \in C^2(\mathbb{R})}}$$

(OBS! I (e), (f) ovan används metod med integrerande faktor från envar. 2.)

4.7)

a) $z(x,y) = x^3 + xy^2$

$z(1+h, 2+k) = z(1,2) + z'_x(1,2)h + z'_y(1,2)k + \text{restterm}$.

$z(1,2) = 1 + 1\cdot 2^2 = 5$, så $(1,2,5)$ ligger på grafen

$z'_x = 3x^2 + y^2$, $z'_x(1,2) = 3 + 4 = 7$

$z'_y = 2xy$, $z'_y(1,2) = 2\cdot 1\cdot 2 = 4$

Tangentplan ges av

$$\underline{\underline{z = 5 + 7h + 4k = 5 + 7(x-1) + 4(y-1)}}$$

b) $z(x,y) = e^{2x} - 1$. $z(0,2) = 1 - 1 = 0$, så $(0,2,0)$ ligger på grafen.

$z'_x = 2e^{2x}$, $z'_x(0,2) = 2$

$z'_y = 0$.

$$\underline{\underline{z = 0 + 2(x-0) + 0(y-2) = 2x}}.$$

c) $y(x,z) = \arcsin xz$, $y(1, 1/2) = \arcsin 1/2 = \pi/6$ ok

$y'_x = \dfrac{1}{\sqrt{1-x^2z^2}} \cdot z$, $y'_x(1, 1/2) = \dfrac{1/2}{\sqrt{1-1/4}} = \dfrac{1}{\sqrt{3}}$

$y'_z = \dfrac{1}{\sqrt{1-x^2z^2}}\, x$, $y'_z(1, 1/2) = \dfrac{1}{\sqrt{1-1/4}} = \dfrac{2}{\sqrt{3}}$

$$y = \frac{\pi}{6} + \frac{1}{\sqrt{3}}(x-1) + \frac{2}{\sqrt{3}}\left(z - \frac{1}{2}\right) =$$

$$= \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}z + \frac{\pi}{6} - \frac{2}{\sqrt{3}}$$

4.8)

$$P(U,R) = \frac{U^2}{R}$$

$P(10,2) = 50$

$P'_U = \frac{2U}{R}$, $P'_U(10,2) = 10$

$P'_R = -\frac{U^2}{R^2}$, $P'_R(10,2) = -25$

$$P(10+h, 2+k) \approx P(10,2) + P'_U(10,2)h + P'_R(10,2)k$$

$$= 50 + 10h - 25k$$

om h, k små, så

a) $P(10.3, 2.1) - P(10,2) \approx 10 \cdot 0.3 - 25 \cdot 0.1 = \underline{\underline{0.5}}$

b) $P(10.3, 2.2) - P(10,2) \approx 10 \cdot 0.3 - 25 \cdot 0.2 = \underline{\underline{-2}}$

4.9)

$$f(x,y) = xy$$

$$f(1+h,\, 2+k) - f(1,2) =$$

$$= (1+h)(2+k) - 2 = 2h + k + hk$$

$$= 2h + k + \frac{hk}{\sqrt{h^2+k^2}} \sqrt{h^2+k^2} \,.$$

$$\frac{hk}{\sqrt{h^2+k^2}} \to 0 \quad \text{då } (h,k) \to (0,0)$$

D.v.s. $df(1,2)(h,k) = 2h + k$.

4.10 a)

$$\frac{f(h,0)-f(0,0)}{h} = \frac{\frac{h^3}{h^2}-0}{h} = 1 \to 1 \text{ då } h \to 0$$

så $f'_x(0,0) = 1$

$$\frac{f(0,k)-f(0,0)}{k} = \frac{\frac{k^4}{k^2}-0}{k} = k \to 0 \text{ då } k \to 0$$

så $f'_y(0,0) = 0$.

$$R(h,k) = \frac{f(h,k)-f(0,0)-1\cdot h-0\cdot k}{\sqrt{h^2+k^2}} =$$

$$= \frac{\frac{h^3+k^4}{h^2+k^2}-0-h}{\sqrt{h^2+k^2}} = \frac{h^3+k^4-h^3-hk^2}{(h^2+k^2)^{3/2}}$$

f är differentierbar i $(0,0)$ om och endast om detta $\to 0$ då $(h,k) \to (0,0)$.

Men längs kurvan $h=k$ får vi

$$R(h,h) = \frac{h^3+h^4-h^3-h^3}{2^{3/2}h^3} = \frac{h-1}{2^{3/2}} \to \frac{-1}{2^{3/2}} \neq 0$$

då $h \to 0$. Alltså ej differentierbar.

Men $\frac{x^3+y^4}{x^2+y^2} = \rho(\cos^3\varphi + \rho\sin^4\varphi) \to 0$ då $\rho \to 0$

så $f(x,y) \to 0$ då $(x,y) \to (0,0)$.

Så $f(x,y)$ är kontinuerlig i $(0,0)$.

(Att f är kontinuerlig i alla andra punkter följer av standardsatser för gränsvärden.)

4.10 b)

$f(x,0) = 1$ för alla x, så

$f'_x(x,0) = 0$ för alla x.

$f(0,y) = 1$ för alla y, så

$f'_y(0,y) = 0$ för alla y.

(Att f'_x, f'_y existerar om $(x,y) \neq (0,0)$ är uppenbart då $f(x,y) = \frac{(x+y)^2}{x^2+y^2}$ där...)

Men

$$f(x,-x) = \frac{0}{2x^2} \to 0 \text{ då } x \to 0,$$

så vi har inte att $f(x,y) \to 1$ då $(x,y) \to (0,0)$.

c) $f(x,0) = 0$ per definition för alla x, och

om $y \neq 0$: $f(0,y) = y^2 \arctan 0 = 0$, så $f(0,y) = 0$ för alla y.

så $f'_x(x,0) = 0$ och $f'_y(0,y) = 0$ för alla x, y.

$$\left| \frac{f(h,k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} \right|$$

$$= \left| \frac{k^2 \arctan(h/k^2)}{\sqrt{h^2+k^2}} \right| \leq |k| \, |\arctan(h/k^2)| \leq \frac{\pi}{2} k$$

$$\to 0 \text{ då } (h,k) \to (0,0)$$

Så f är differentierbar i $(0,0)$

Men för $y \neq 0$ gäller

$$f'_x = y^2 \cdot \frac{1}{1+\left(\frac{x}{y^2}\right)^2} \cdot \frac{1}{y^2} = \frac{1}{1+\frac{x^2}{y^4}} = \frac{y^4}{x^2+y^4}$$

så $f'_x(0,y) = 1 \not\to 0$ då $y \to 0$.

H. 11

$$\begin{cases} z'_x = y e^{x^2y^4} \\ z'_y = x e^{x^2y^4} \end{cases}$$

$$z''_{xy} = (z'_x)'_y = \frac{\partial}{\partial y}(ye^{x^2y^4}) = e^{x^2y^4} + 4y^4x^2e^{x^2y^4}$$

$$z''_{yx} = (z'_y)'_x = \frac{\partial}{\partial x}(xe^{x^2y^4}) = e^{x^2y^4} + 2x^2y^4e^{x^2y^4}.$$

$z''_{xy} \neq z''_{yx}$ så kan ej finnas $z \in C^2$ som uppfyller detta.