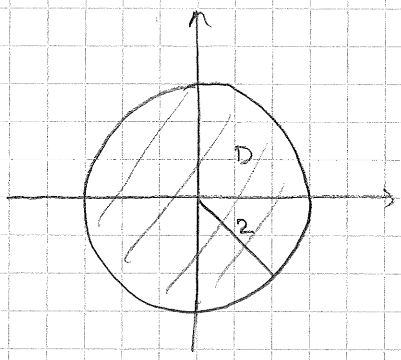
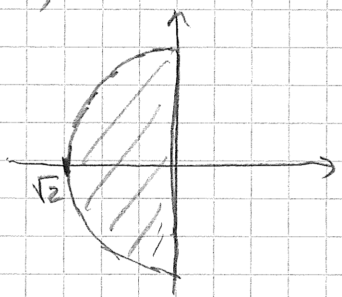
6.9)

a)



$$\iint_D e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\left(\int_0^2 e^{\rho^2}\rho\,d\rho\right)d\varphi = 2\pi\left[\frac{e^{\rho^2}}{2}\right]_0^2 = \underline{\underline{\pi(e^4-1)}}.$$

b)



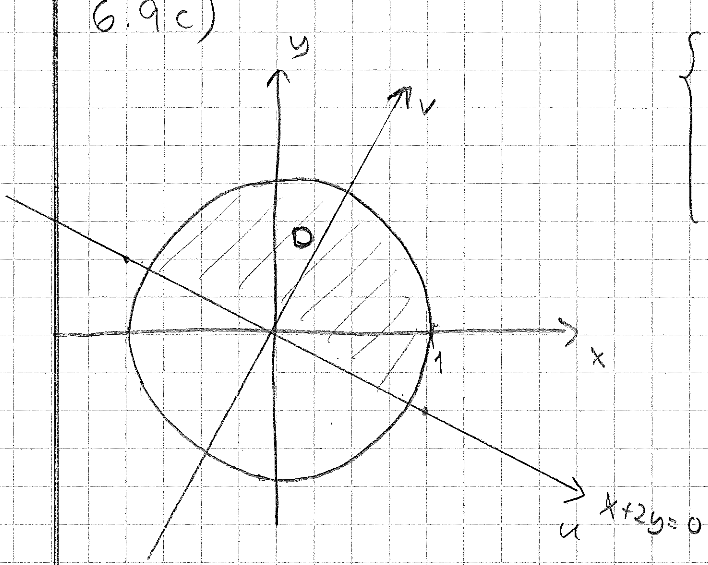
$\pi/2 \le \varphi \le 3\pi/2 \qquad 0 \le \rho \le \sqrt{2}$

$$\iint_D \frac{x}{1+(x^2+y^2)^{3/2}}\,dx\,dy = \int_{\pi/2}^{3\pi/2}\left(\int_0^{\sqrt{2}} \frac{\rho\cos\varphi}{1+\rho^3}\,\rho\,d\rho\right)d\varphi =$$

$$= \int_{\pi/2}^{3\pi/2}\cos\varphi\,d\varphi \cdot \int_0^{\sqrt{2}}\frac{\rho^2}{1+\rho^3}\,d\rho = \Big[\sin\varphi\Big]_{\pi/2}^{3\pi/2} \cdot \left[\frac{1}{3}\ln(1+\rho^3)\right]_0^{\sqrt{2}} =$$

$$= \underline{\underline{-\frac{2}{3}\ln(1+2\sqrt{2})}}$$

6.9 c)



$$\begin{cases} u = \dfrac{2x-y}{\sqrt{5}} \\[2ex] v = \dfrac{x+2y}{\sqrt{5}} \end{cases}$$

$$\frac{d(u,v)}{d(x,y)} = \frac{1}{\sqrt{5}^2}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 1.$$

$x^2+y^2 = u^2+v^2$, så

$\Omega = \{(u,v): \ u^2+v^2 \le 1, \ v \ge 0\}$

$$\iint_D (x+2y)\,dx\,dy = \iint_\Omega \sqrt{5}\,v \left|\frac{d(x,y)}{d(u,v)}\right| du\,dv =$$

$$= \int_0^\pi \left(\int_0^1 \sqrt{5}\rho\sin\varphi\ \rho\,d\rho \right) d\varphi = \sqrt{5}\int_0^\pi \sin\varphi\,d\varphi \cdot \int_0^1 \rho^2\,d\rho = \underline{\underline{\frac{2\sqrt{5}}{3}}}$$

d)

$$\iint_D |x+y|\,dx\,dy = \Bigg/ \ u = \frac{x+y}{\sqrt{2}} \quad \frac{d(u,v)}{d(x,y)} = \frac{1}{\sqrt{2}^2}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1$$

$$v = \frac{-x+y}{\sqrt{2}}$$

$$D \mapsto \Omega = \{(u,v): \ u^2+v^2 \le 1\} \ \Bigg/$$

$$= \iint_\Omega |\sqrt{2}u| \left|\frac{d(x,y)}{d(u,v)}\right| du\,dv = \sqrt{2}\int_0^{2\pi} \left(\int_0^1 |\rho\cos\varphi|\,\rho\,d\rho \right) d\varphi =$$

$$= \sqrt{2}\int_0^{2\pi} |\cos\varphi| \cdot \int_0^1 \rho^2\,d\rho = 4\sqrt{2}\cdot\int_0^{\pi/2} \cos\varphi\,d\varphi \cdot \frac{1}{3} = \underline{\underline{\frac{4\sqrt{2}}{3}}}.$$

6.10)

a) $D: 0 \le x+2y \le 1, \; -1 \le 2x+y \le 1.$

Med $u = x+2y$, $v = 2x+y$ ges området av

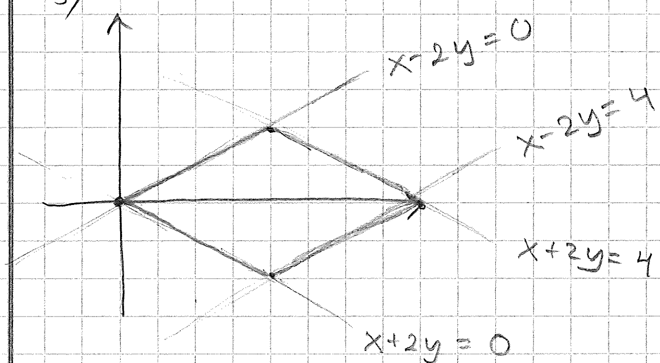
$0 \le u \le 1, \; -1 \le v \le 1.$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\iint_D (x+2y)\cos(2x+y)\,dx\,dy = \int_0^1 \left(\int_{-1}^1 u\cos v \left| \frac{d(x,y)}{d(u,v)} \right| dv \right) du =$$

$$= \frac{1}{3} \int_0^1 u\,du \cdot \int_{-1}^1 \cos v\,dv = \frac{1}{3} \cdot \frac{1}{2} \cdot (\sin 1 - \sin(-1)) = \underline{\underline{\frac{\sin 1}{3}}}$$

b)



$$\begin{cases} u = x+2y \\ v = x-2y \end{cases}$$. Området ges av $0 \le u \le 4, \; 0 \le v \le 4.$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -4.$$

$$\iint_D (x+2y)\exp(x-2y)\,dx\,dy = \int_0^4 \left(\int_0^4 u \exp v \left| \frac{d(x,y)}{d(u,v)} \right| du \right) dv$$

$$= \frac{1}{4} \int_0^4 u\,du \cdot \int_0^4 \exp v\,dv = \underline{\underline{2(e^4-1)}}$$

6.10)

c) Med $u = x+y$, $v = 2x-4y$ ges området av

$1 \le u \le v \le 3$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} = -6$$

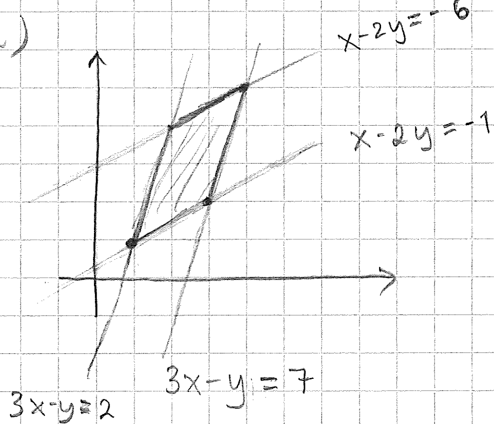
$$\iint_D \frac{x-y+1}{x+y}\,dxdy = \Big/ \begin{matrix} u+v = 3x-3y \\ dxdy = \frac{1}{6}dudv \end{matrix} \Big/ = \frac{1}{6}\int_1^3 \left(\int_1^v \frac{\frac{u+v}{3}+1}{u}\,du \right) dv$$

$$= \frac{1}{18}\int_1^3 \left(\int_1^v \left(1 + \frac{v+3}{u}\right) du \right) dv = \frac{1}{18}\int_1^3 \Big[u + (v+3)\ln u\Big]_{u=1}^{v} dv$$

$$= \frac{1}{18}\int_1^3 (v + (v+3)\ln v - 1)\,dv = \frac{1}{18}\left[\frac{v^2}{2} + \frac{v^2}{2}\ln v - \frac{v^2}{4} + 3v\ln v - 3v - v\right]_1^3$$

$$= \underline{\underline{\frac{3\ln 3}{4} - \frac{1}{3}}}$$

d)



$u = x-2y$, $v = 3x-y$

Området ges av

$-6 \le u \le -1$, $2 \le v \le 7$.

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$$

$$dxdy = \frac{1}{7}dudv$$

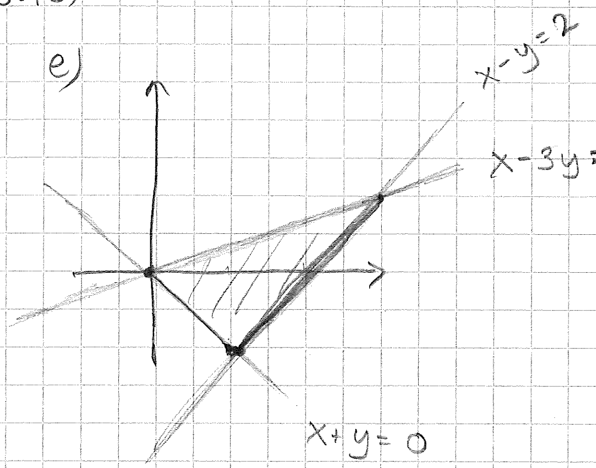
$$\iint_D x^2\,dxdy = \Big/ 2v-u = 6x-2y-x+2y = 5x \Big/ = \frac{1}{5}\int_2^7 \left(\int_{-6}^{-1} \left(\frac{2v-u}{5}\right)^2 du \right) dv$$

$$= \frac{1}{5^3}\int_2^7 \left(\int_{-6}^{-1} (4v^2 - 4uv + u^2)\,du \right) dv = \frac{1}{5^3}\int_2^7 \left[4v^2u - 2u^2v + \frac{u^3}{3}\right]_{-6}^{-1} dv$$

$$= \frac{1}{5^3}\int_2^7 \left(-4v^2 - 2v - \frac{1}{3} + 24v^2 + 72v + 72\right) dv = \ldots = \frac{100}{3}$$

6.10)

e)

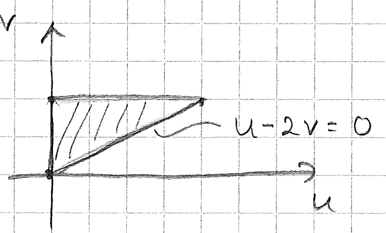


Integranden $\frac{1}{(1+x^2-y^2)^2} = \frac{1}{(1+(x+y)(x-y))^2}$

antyder tillsammans med bilden ovan att

$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$ är lämpligt byte.

I uv-planet får punkterna som i xy-planet har koordinater (0,0), (1,-1) och (3,1) koord. (0,0), (0,2), (4,2)



$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

$dxdy = \frac{1}{2} du dv.$

$$\iint_D \frac{1}{(1+x^2-y^2)^2} dxdy = \int_0^2 \left(\int_0^{2v} \frac{1}{(1+uv)^2} \frac{1}{2} du \right) dv$$

$$= \frac{1}{2}\int_0^2 \left[-\frac{1}{v(1+uv)} \right]_0^{2v} dv = \frac{1}{2} \int_0^2 \left(\frac{1}{v} - \frac{1}{v(1+2v^2)} \right) dv$$

$$= \frac{1}{2} \int_0^2 \left(\frac{1}{v} - \frac{1}{v} + \frac{2v}{1+2v^2} \right) dv = \frac{1}{4} \left[\ln(1+2v^2) \right]_0^2 =$$

$$= \frac{1}{4} \ln(9) = \underline{\underline{\frac{\ln 3}{2}}}$$

6.11)

a)

Med $x = 2\rho\cos\varphi$, $y = 3\rho\sin\varphi$ får vi

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4\rho^2\cos^2\varphi}{4} + \frac{9\rho^2\sin^2\varphi}{9} = \rho^2 \le 1.$$

Så i $\rho\varphi$-planet ges området av $0 \le \varphi \le 2\pi$, $0 \le \rho \le 1$.

$$\frac{d(x,y)}{d(\rho,\varphi)} = \begin{vmatrix} 2\cos\varphi & -2\rho\sin\varphi \\ 3\sin\varphi & 3\rho\cos\varphi \end{vmatrix} = 6\rho$$

D.v.s. $dx\,dy = 6\rho\,d\rho\,d\varphi$.

$$\iint_D (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\left(\int_0^1 (4\rho^2\cos^2\varphi + 9\rho^2\sin^2\varphi)6\rho\,d\rho\right)d\varphi =$$

$$= 6\int_0^{2\pi}(4\cos^2\varphi + 9\sin^2\varphi)\,d\varphi \cdot \int_0^1 \rho^3\,d\rho = \ldots = \frac{39\pi}{2}.$$

b)

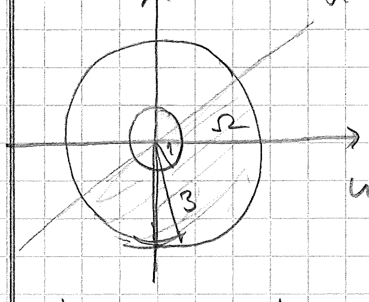
Om vi först byter koordinater $\begin{cases} u = x \\ v = 3y \end{cases}$

får vi $dx\,dy = \frac{1}{3}du\,dv$, och att

området ges av $1 \le u^2+v^2 \le 9$, $u \ge v$.

$$\iint_D x^3\,dx\,dy = \iint_\Omega u^3\frac{1}{3}\,du\,dv = \frac{1}{3}\int_{-\frac{3\pi}{4}}^{\pi/4}\left(\int_1^3 \rho^3\cos^3\varphi\,\rho\,d\rho\right)d\varphi$$

$$= \frac{1}{3}\int_{-\frac{3\pi}{4}}^{\pi/4}\cos^3\varphi\,d\varphi \cdot \int_1^3 \rho^4\,d\rho =$$



$$= \frac{1}{3}\int_{-\frac{3\pi}{4}}^{\pi/4}\cos\varphi(1-\sin^2\varphi)\,d\varphi \cdot \left[\frac{\rho^5}{5}\right]_1^3$$

$$= \frac{242}{15}\left[\sin\varphi - \frac{\sin^3\varphi}{3}\right]_{-3\pi/4}^{\pi/4}$$

I pol. koord. ges Ω av $-\frac{3\pi}{4} \le \varphi \le \frac{\pi}{4}$, $1 \le \rho \le 3$

$$= \frac{242}{15}\left(\frac{2}{\sqrt{2}} - \frac{2}{2\sqrt{2}\cdot 3}\right) = \underline{\underline{\frac{121\sqrt{2}}{9}}}$$

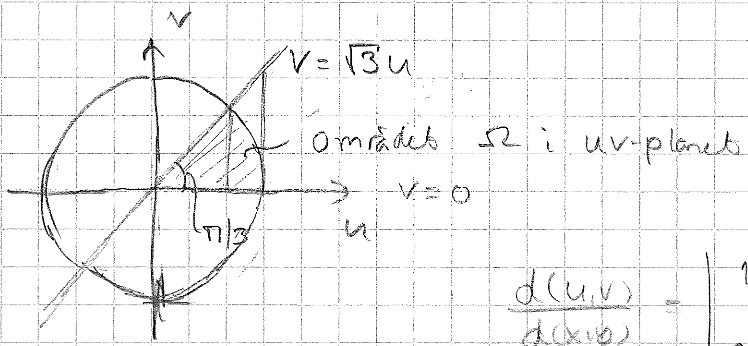
6.11)

c)

$$x^2 + 2xy + 4y^2 = (x+y)^2 + 3y^2 \le 1 \qquad x \ge 0,\ y \ge 0$$

$$\begin{cases} u = x+y \\ v = \sqrt{3}\,y \end{cases} \Leftrightarrow \begin{cases} x+y = u \\ y = \frac{1}{\sqrt{3}} v \end{cases} \Leftrightarrow \begin{cases} x = u - \frac{1}{\sqrt{3}} v \\ y = \frac{1}{\sqrt{3}} v \end{cases}$$

I uv-planet ges området av $u^2+v^2 \le 1$, $u - \frac{1}{\sqrt{3}} v \ge 0$, $\frac{1}{\sqrt{3}} v \ge 0$



$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = \sqrt{3}$$

$$\iint_D x\,dx\,dy = \iint_\Omega \left(u - \frac{v}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\,du\,dv =$$

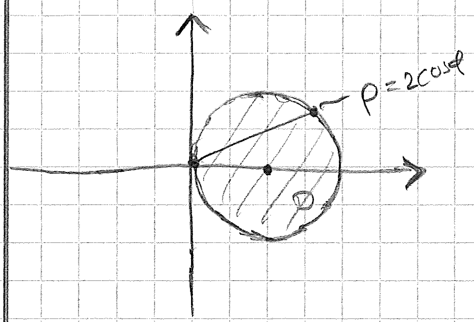
$$= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \left(\int_0^1 \left(\rho\cos\varphi - \frac{\rho\sin\varphi}{\sqrt{3}}\right)\rho\,d\rho \right) d\varphi =$$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \left(\cos\varphi - \frac{\sin\varphi}{\sqrt{3}}\right) d\varphi \cdot \int_0^1 \rho^2\,d\rho = \frac{1}{3\sqrt{3}} \left[\sin\varphi + \frac{\cos\varphi}{\sqrt{3}}\right]_0^{\pi/3} =$$

$$= \frac{1}{3\sqrt{3}} \left(\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \right) = \frac{1}{6} - \frac{1}{18} = \underline{\underline{\frac{1}{9}}}$$

6.12)

$$x^2+y^2 \le 2x \iff (x-1)^2+y^2 \le 1$$



I polära koordinater

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$ ges D av

$$(\rho\cos\varphi - 1)^2 + \rho^2\sin^2\varphi =$$

$$= \rho^2 - 2\rho\cos\varphi + 1 \le 1$$

$$\iff 0 \le \rho \le 2\cos\varphi, \quad -\frac{\pi}{2} \le \varphi \le \pi/2$$

Om vi å andra sidan gör ett linjärt variabelbyte $u = x-1$, $v = y$ får vi vanliga enhetscirkeln i uv-planet; $u^2+v^2 \le 1$.

a) $$\iint_D (x^2+y^2)\,dx\,dy = \left/ \begin{array}{lcl} x-1=u & \iff & x = 1+u \\ y = v & & y = v \\ dx\,dy = du\,dv & & \end{array} \right/ =$$

$$= \iint_{u^2+v^2\le 1} \left((1+u)^2+v^2\right)du\,dv = \int_0^{2\pi}\left(\int_0^1 (\rho^2 + 2\rho\cos\varphi + 1)\rho\,d\rho\right)d\varphi$$

$$= \int_0^{2\pi}\left[\frac{\rho^4}{4} + \frac{2\rho^3}{3}\cos\varphi + \frac{\rho^2}{2}\right]_{\rho=0}^{1} d\varphi = \int_0^{2\pi}\left(\frac{1}{4} + \frac{2\cos\varphi}{3} + \frac{1}{2}\right)d\varphi$$

$$= \left[\frac{3\varphi}{4} + \frac{2\sin\varphi}{3}\right]_0^{2\pi} = \underline{\underline{\frac{3\pi}{2}}}$$

b) $$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos\varphi} \rho\cdot\rho\,d\rho\right)d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{\rho^3}{3}\right]_0^{2\cos\varphi} d\varphi = \frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos^3\varphi\,d\varphi = \frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos\varphi(1-\sin^2\varphi)\,d\varphi =$$

$$= \frac{8}{3}\left[\sin\varphi - \frac{\sin^3\varphi}{3}\right]_{-\pi/2}^{\pi/2} = \frac{8}{3}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \underline{\underline{\frac{32}{9}}}.$$

6.13) $1 \le xy \le 2, \quad 0 \le x \le y \le 2x$

$\Leftrightarrow \quad 1 \le xy \le 2, \quad 0 \le x, \quad xy \le y^2 \le 2xy$

Med $u = xy$, $v = y^2$ avbildas D injektivt på området $1 \le u \le 2$, $u \le v \le 2u$.

OBS! Valet av dessa koordinater är anpassat dels efter området, och dels efter integranden $y^2 \sin y^2$.

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} y & x \\ 0 & 2y \end{vmatrix} = 2y^2 = 2v \quad , \quad dxdy = \frac{1}{2v}\,dudv.$$

$$\iint_D y^2 \sin y^2 \, dxdy = \int_1^2 \left(\int_u^{2u} v \sin v \, \frac{1}{2v} \, dv \right) du =$$

$$= \frac{1}{2} \int_1^2 \Big[-\cos v \Big]_{v=u}^{2u} dv = \frac{1}{2} \int_1^2 (\cos u - \cos 2u)\, du =$$

$$= \frac{1}{2} \left[\sin u - \frac{\sin 2u}{2} \right]_1^2 = \frac{1}{2} \left(\sin 2 - \frac{\sin 4}{2} - \sin 1 + \frac{\sin 2}{2} \right) =$$

$$= \frac{3\sin 2 - \sin 4 - 2\sin 1}{4}$$