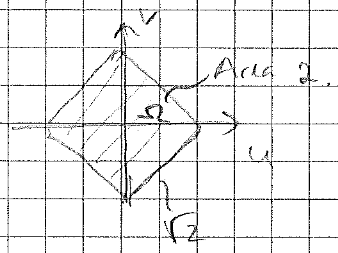
6.29)

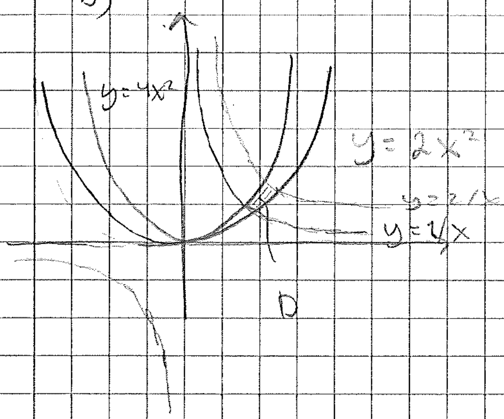
a) $\begin{cases} u = x + 2y \\ v = 3x - y \end{cases}$ överför området på $|u| + |v| \le 1$.

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7$$



$$\iint_D dxdy = \iint_\Omega \frac{1}{7}\, dudv = \underline{\underline{\frac{2}{7}}}$$

b)



$\begin{cases} u = y/x^2 \\ v = xy \end{cases}$

$y = 4x^2 \Leftrightarrow y/x^2 = u = 4$

$y = 2x^2 \Leftrightarrow$ ~~$y/2x^2 \quad u=2$~~ $y/x^2 = u = 2$

$xy = 1 \Leftrightarrow v = 1$, $xy = 2 \Leftrightarrow v = 2$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ y & x \end{vmatrix} = \frac{-2y}{x^2} - \frac{y}{x^2} = \frac{-3y}{x^2} = 3u$$

D.v.s. $dxdy = \frac{1}{3u}\, dudv$.

$$\iint_D dxdy = \int_1^2 \left(\int_2^4 \frac{1}{3u}\, du \right) dv = \underline{\ln 2/3}$$

6.30)

a)

$X = au,\ y = bv,\ z = cw$ ger att

D på $u^2+v^2+w^2 \leq 1$, som har volym $\frac{4\pi}{3}$ i uvw-rummet

$$\frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Så Volym$(D) = \frac{4\pi abc}{3}$.

b)

$$3\left(x+\frac{z}{3}\right)^2 - \frac{z^2}{3} + 2y^2 + z^2 - 2yz =$$

$$= 3\left(x+\frac{z}{3}\right)^2 + 2\left(y-\frac{z}{2}\right)^2 - \frac{z^2}{2} - \frac{z^2}{3} + z^2 =$$

$$= 3\left(x+\frac{z}{3}\right)^2 + 2\left(y-\frac{z}{2}\right)^2 + \frac{z^2}{6} \leq 1$$

$$\begin{cases} u = \left(x+\frac{z}{3}\right) \\ v = y - \frac{z}{2} \\ w = z \end{cases} \qquad 3u^2 + 2v^2 + \frac{w^2}{6} \leq 1$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Enligt (a) har $3u^2+2v^2+\frac{w^2}{6} \leq 1$ area

$\frac{4\pi abc}{3}$ då $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{2}}$, $c = \sqrt{6}$,

D.v.s. $\frac{4\pi \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \sqrt{6}}{3} = \underline{\underline{\frac{4\pi}{3}}}$.

6.31)

$$\begin{cases} u = 2x+y+z \\ v = x+2y+z \\ w = x+y+2z \end{cases}$$

$$u+v+w = 4x+4y+4z.$$

Så området avbildas på området Ω i uvw-rummet som avgränsas av planen $u=0$, $v=0$, $w=0$ och $u+v+w=16$.

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 4 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4.$$

Den sökta volymen ges alltså av

$$\frac{1}{4}\iiint_\Omega du\,dv\,dw = \frac{1}{4}\int_0^{16}\left(\int_0^{16-u}\left(\int_0^{16-u-v} dw\right)dv\right)du =$$

$$= \frac{1}{4}\int_0^{16}\left(\int_0^{16-u}(16-u-v)\,dv\right)du = \frac{1}{4}\int_0^{16}\left[(16-u)v - \frac{v^2}{2}\right]_0^{16-u} du$$

$$= \frac{1}{4}\int_0^{16}\frac{(16-u)^2}{2}\,du = \frac{1}{4}\left[-\frac{(16-u)^3}{6}\right]_0^{16} = \frac{1}{4}\cdot\frac{16^3}{6} = \frac{2\cdot 16^2}{3} = \frac{512}{3}$$

6.32)

a) $9x^2+4y^2 \le 36$: Ω, projektion på xy-planet.

$z = 10-x-y$, $z = x+y-10$

Eftersom $x^2 \le \frac{36}{9} = 4 \iff -2 \le x \le 2$

$y^2 \le \frac{36}{4} = 9 \iff -3 \le y \le 3$

ser vi att det första av dessa plan ligger

över det andra, d.v.s. $x+y-10 \le z \le 10-x-y$.

Volymen ges av

$$\iint_\Omega \left(\int_{x+y-10}^{10-x-y} dz \right) dx\,dy = \iint_\Omega (20-2x-2y)\,dx\,dy =$$

$$= \left/ \begin{array}{l} x = \frac{\rho}{3}\cos\varphi \quad \Omega \text{ ges av} \\ y = \frac{\rho}{2}\sin\varphi \quad \rho^2 \le 36, \text{ dvs } 0 \le \rho \le 6,\ 0 \le \varphi \le 2\pi. \end{array} \quad \frac{d(x,y)}{d(\rho,\varphi)} = \frac{1}{6}\rho \right/$$

$$= \int_0^{2\pi} \left(\int_0^6 \left(20 - 2\frac{\rho}{3}\cos\varphi - 2\frac{\rho}{2}\sin\varphi\right)\frac{1}{6}\rho\,d\rho \right) d\varphi =$$

$$= \frac{1}{6}\int_0^{2\pi} \left[10\rho^2 - \frac{2\rho^3}{9}\cos\varphi - \frac{2\rho^3}{6}\sin\varphi \right]_{\rho=0}^{6} d\varphi =$$

$$= \frac{1}{6}\int_0^{2\pi} \left(10\cdot 6^2 - \frac{2\cdot 6^3}{9}\cos\varphi - \frac{2\cdot 6^3}{6}\sin\varphi \right) d\varphi =$$

$$= \Big[60\varphi - 8\sin\varphi + 12\cos\varphi \Big]_0^{2\pi} = \underline{\underline{120\pi}}$$

6.32)

b)

$$x^2+y^2=1-2x-2y \iff (x+1)^2+(y+1)^2=3$$

Området har alltså projektion som är cirkeln med centrum i $(-1,-1)$ och radie $\sqrt{3}$.

$0 \le z \le 1-2x-2y$ för varje (x,y) i denna projektion

Volymen ges av

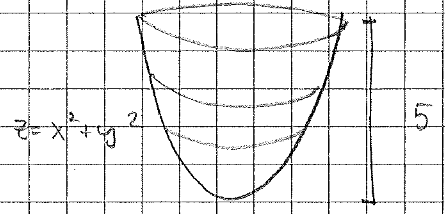
$$V=\iint_{\Omega}\left(\int_{x^2+y^2}^{1-2x-2y} dz\right)dxdy$$ där Ω är projektionen på xy-planet.

$$V=\iint_{\Omega}(1-2x-2y-x^2-y^2)\,dxdy=\iint_{\Omega}(3-(x+1)^2-(y+1)^2)$$

$$=\left/\begin{matrix} x+1=\rho\cos\varphi \\ y+1=\rho\sin\varphi \end{matrix}\right/=\int_0^{2\pi}\left(\int_0^{\sqrt{3}}(3-\rho^2)\rho\,d\rho\right)d\varphi=$$

$$=2\pi\left[\frac{3\rho^2}{2}-\frac{\rho^4}{4}\right]_0^{\sqrt{3}}=2\pi\left(\frac{9}{2}-\frac{9}{4}\right)=\underline{\underline{\frac{9\pi}{2}}}\,!$$

6.36)



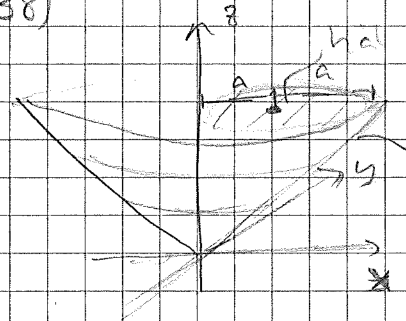
$D = \{(x,y,z) : 0 \le z \le 5,\ x^2+y^2 \le z\}$.

$D_z = \{(x,y) : x^2+y^2 \le z\}$

D_z har area $\pi \cdot \sqrt{z}^2 = \pi z$

$$\iiint_D dx\,dy\,dz = \int_0^5 \left(\iint_{D_z} dx\,dy \right) dz = \int_0^5 \pi z\,dz = \underline{\underline{\frac{25\pi}{2}}}.$$

6.38)



$z = \frac{h}{2a}\sqrt{x^2+y^2}$.

Konen ges av olikheterna $x^2+y^2 \le (2a)^2 = 4a^2$, $\frac{h}{2a}\sqrt{x^2+y^2} \le z \le h$.

Så den har volym

$$\iint_{B(0,2a)} \left(\int_{\frac{h\sqrt{x^2+y^2}}{2a}}^{h} dz \right) dx\,dy = \int_0^{2\pi} \left(\int_0^{2a} \left(h - \frac{h\rho}{2a}\right) \rho\, d\rho \right) d\varphi = \ldots = \frac{4\pi h a^2}{3}.$$

Hålet ges av olikheterna $(x-a)^2+y^2 \le a^2$ $\frac{h}{2a}\sqrt{x^2+y^2} \le z \le h$.

I polära koordinater i xy-planet ges området:

$(x-a)^2+y^2 \le a^2 \Leftrightarrow (\rho\cos\varphi - a)^2 + \rho^2\sin^2\varphi = \rho^2 - 2a\rho\cos\varphi + a^2 \le a^2$

$\Leftrightarrow 0 \le \rho \le 2a\cos\varphi$, $-\pi/2 \le \varphi \le \pi/2$.

Volymen av hålet blir nu

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^{2a\cos\varphi} \left(\int_{\frac{h\rho}{2a}}^{h} dz \right) \rho\, d\rho \right) d\varphi = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2a\cos\varphi} \left(h - \frac{h\rho}{2a}\right) \rho\, d\rho \right) d\varphi$$

$$= \ldots = \left(\pi - \frac{16}{9}\right) h a^2.$$

Så volym kon - volym hål $= \frac{4\pi h a^2}{3} - \left(\pi - \frac{16}{9}\right) h a^2 = \underline{\underline{\left(\frac{\pi}{3} + \frac{16}{9}\right) a^2 h}}$

6.39) a) Volym av ett halvklot är $\frac{2\pi \cdot R^3}{3}$

Så massa blir $\frac{2\pi R^3 \rho}{3}$

b) $\rho(x,y,z) = k\sqrt{x^2+y^2+z^2}$.

$$\text{Massa} = \iiint_D \rho(x,y,z) = k\int_0^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^R r\cdot r^2\sin\theta\, dr\right)d\phi\right)d\theta$$

$$= k\int_0^{\pi/2}\sin\theta\, d\theta \cdot 2\pi\cdot\int_0^R r^3\, dr = \frac{2\pi kR^4}{4} = \underline{\underline{\frac{\pi kR^4}{2}}}$$

6.40) Silon ger i koordinaterne

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$$

av $0 \le z \le 8$, $0 \le \rho \le 3$, $0 \le \phi \le 2\pi$

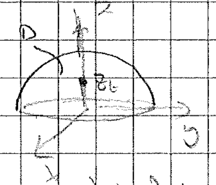
$dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz$.

$$\int_0^8\left(\int_0^{2\pi}\left(\int_0^3 (10-z)^2(4-\rho)\rho\, d\rho\right)d\phi\right)dz = 2\pi\int_0^8 (10-z)^2\, dz\cdot\int_0^3(4-\rho)\rho\, d\rho =$$

$$= 2\pi\left[\frac{-(10-z)^3}{3}\right]_0^8\cdot\left[2\rho^2-\frac{\rho^3}{3}\right]_0^3 = 18\pi\left(\frac{1000}{3}-\frac{8}{3}\right) = \underline{\underline{5952\pi\, kg}}.$$

6.4

6.41) a)



$$z_t = \frac{\iiint_D z\,dx\,dy\,dz}{\iiint_D dx\,dy\,dz} = \frac{\int_0^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^R r\cos\theta \cdot r^2\sin\theta\,dr\right)d\varphi\right)d\theta}{\frac{2\pi R^3}{3}} =$$

$$= \frac{2\pi\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \int_0^R r^3\,dr}{\frac{2\pi R^3}{3}} = \frac{3}{R^3}\cdot\int_0^{\pi/2}\frac{\sin 2\theta}{2}\,d\theta\cdot\frac{R^4}{4} =$$

$$= \frac{3R}{4}\left[-\frac{\cos 2\theta}{4}\right]_0^{\pi/2} = \frac{3R}{8}$$

SVAR: $\left(0, 0, \frac{3R}{8}\right)$.

b)

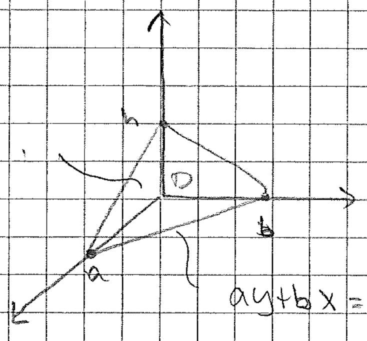
$$z_t = \frac{\iiint_D z\,k\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz}{\iiint_D k\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz} =$$

$$= \frac{k\int_0^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^R r\cos\theta\, r\cdot r^2\sin\theta\,dr\right)d\varphi\right)d\theta}{k\int_0^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^R r\cdot r^2\sin\theta\,dr\right)d\varphi\right)d\theta} =$$

$$= \frac{\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \int_0^R r^4\,dr}{\int_0^{\pi/2}\sin\theta\,d\theta\cdot\int_0^R r^3\,dr} = \frac{4R^5}{5R^4}\cdot\frac{\frac{1}{2}}{1} = \frac{2R}{5}$$

SVAR: $\left(0, 0, \frac{2R}{5}\right)$.

6.42)



$D = \{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b - \frac{bx}{a},$
$0 \le z \le h - \frac{h}{a}x - \frac{h}{b}y\}$

$ay + bx = ab \Leftrightarrow y = b - \frac{bx}{a}$

$z = \alpha x + \beta y + \gamma$ som går genom $(a,0,0), (0,b,0)$ och $(0,0,h)$

ger $h = \gamma,\quad 0 = \alpha a + \gamma,\ 0 = \beta b + \gamma,$

d.v.s. $\alpha = -\frac{h}{a},\ \beta = -\frac{h}{b}$.

$$\iiint_D z\,dx\,dy\,dz = \int_0^a \left(\int_0^{b-\frac{bx}{a}} \left(\int_0^{h-\frac{hx}{a}-\frac{hy}{b}} z\,dz \right) dy \right) dx =$$

$$= \frac{1}{2} \int_0^a \left(\int_0^{b-\frac{bx}{a}} \left(h - \frac{hx}{a} - \frac{hy}{b} \right)^2 dy \right) dx$$

$$= \frac{1}{2} \int_0^a \left[-\frac{b}{h} \frac{\left(h - \frac{hx}{a} - \frac{hy}{b}\right)^3}{3} \right]_{y=0}^{b-\frac{bx}{a}} dx =$$

$$= \frac{1}{2} \int_0^a \frac{b}{3h} \left(h - \frac{hx}{a} \right)^3 dx = \frac{1}{2} \cdot \frac{b}{3h} \left[\frac{a}{4h} \left(h - \frac{hx}{a} \right)^4 \right]_0^a$$

$$= \frac{abh^2}{24}$$

$$\iiint_D dx\,dy\,dz = \int_0^a \left(\int_0^{b-\frac{bx}{a}} \left(\int_0^{h-\frac{hx}{a}-\frac{hy}{b}} dz \right) dy \right) dx =$$

$$= \int_0^a \left(\int_0^{b-\frac{bx}{a}} \left(h - \frac{hx}{a} - \frac{hy}{b} \right) dy \right) dx = \int_0^a \left[\frac{-b}{2h} \left(h - \frac{hx}{a} - \frac{hy}{b} \right)^2 \right]_{y=0}^{b-\frac{bx}{a}} dx$$

$$= \int_0^a \frac{b}{2h} \left(h - \frac{hx}{a} \right)^2 dx = \left[\frac{-ab}{6h^2} \left(h - \frac{hx}{a} \right)^3 \right]_0^a = \frac{abh}{6}$$

$$z_t = \frac{abh^2}{24} \bigg/ \frac{abh}{6} = \underline{\underline{\frac{h}{4}}}.$$

6.33)

a) Eftersom $x, y, z \geq 0$, gäller att

$x + y^2 + z \leq 1$ tvingar $x \leq 1, y^2 \leq 1, z \leq 1 \Leftrightarrow x \leq 1, y \leq 1, z \leq 1$.

$0 \leq V \leq 1$.

b) Projektion på xz-planet $x \geq 0, z \geq 0, x + z \leq 1$

$0 \leq y^2 \leq 1 - x - z \Leftrightarrow$ / eftersom $y \geq 0$ / $\Leftrightarrow 0 \leq y \leq \sqrt{1-x-z}$

$$\iiint_D dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{\sqrt{1-x-z}} dy \right) dz \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \sqrt{1-x-z}\, dz \right) dx = \int_0^1 \left[-\frac{2}{3}(1-x-z)^{3/2} \right]_{z=0}^{1-x} dx$$

$$= \int_0^1 \frac{2}{3}(1-x)^{3/2} dx = \left[-\frac{2}{3} \cdot \frac{2}{5} (1-x)^{5/2} \right]_0^1 = \underline{\underline{\frac{4}{15}}}$$

6.34) $D = \{(x,y,z) : z \geq y^2, y \geq x^2, x \geq z^2\} =$

$= \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, z^2 \leq x \leq \sqrt{y} \leq \sqrt[4]{z}\}$

Projektion på xz-planet $\tilde{D}$ ges av $0 \leq z \leq 1$, $z^2 \leq x \leq \sqrt[4]{z}$,

om för varje $(x,z) \in \tilde{D}$ uppfyller y

$x^2 \leq y \leq \sqrt{z}$.

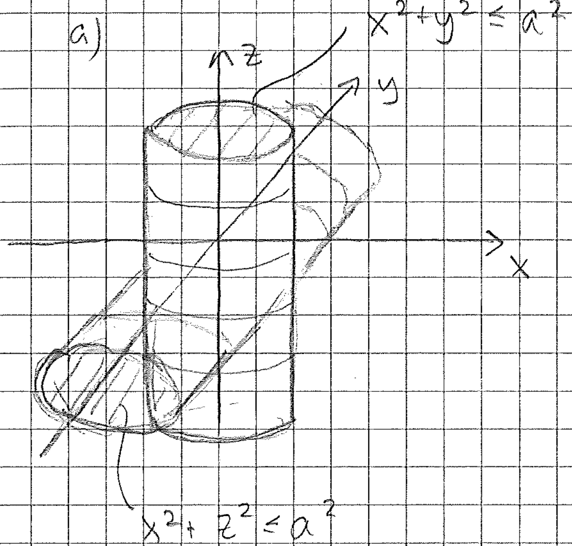
$$\int_0^1 \left(\int_{z^2}^{\sqrt[4]{z}} \left(\int_{x^2}^{\sqrt{z}} dy \right) dx \right) dz = \int_0^1 \left(\int_{z^2}^{\sqrt[4]{z}} (\sqrt{z} - x^2)\, dx \right) dz$$

$$= \int_0^1 \left[\sqrt{z}\,x - \frac{x^3}{3} \right]_{x=z^2}^{\sqrt[4]{z}} dz = \int_0^1 \left(\frac{2z^{3/4}}{3} - z^{5/2} + \frac{z^6}{3} \right) dz =$$

$$= \left[\frac{2}{3} \cdot \frac{4}{7} \cdot z^{7/4} - \frac{2}{7} \cdot z^{7/2} + \frac{z^7}{21} \right]_0^1 = \frac{8}{21} - \frac{2}{7} + \frac{1}{21} = \frac{3}{21} = \underline{\underline{\frac{1}{7}}}$$

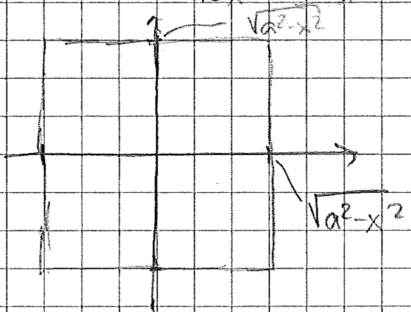
6.35)

a)



Projektion på x-axeln $-a \le x \le a$.

Snitt $D_x = \{(y,z) : -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},\ -\sqrt{a^2-x^2} \le z \le \sqrt{a^2-x^2}\}$.



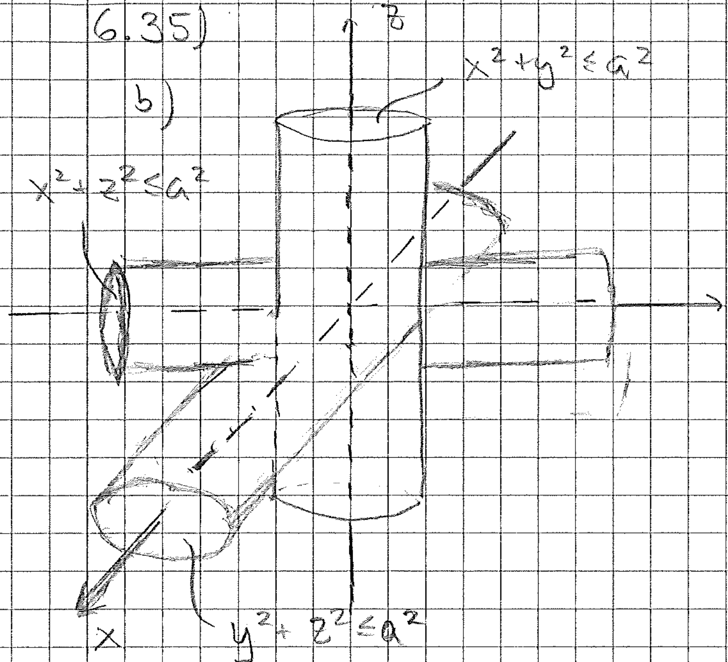
D_x är kvadrat med sidlängd $2\sqrt{a^2-x^2}$. Area $4(a^2-x^2)$.

$$\iiint_D dx\,dy\,dz = \int_{-a}^{a}\left(\iint_{D_x} dy\,dz\right)dx = \int_{-a}^{a} 4(a^2-x^2)\,dx =$$

$$= 4\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \underline{\underline{\frac{16a^3}{3}}}$$

6.35)

b)



$D = \{(x,y,z) : \; x^2+y^2 \le a^2, \; x^2+z^2 \le a^2, \; y^2+z^2 \le a^2\}$

Projektion $\tilde{D} = \{(x,y) : \; x^2+y^2 \le a^2\}$

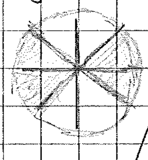
Gränserna för z ges av att både

$x^2+z^2 \le a^2$ och $y^2+z^2 \le a^2$. D.v.s

$-\min(\sqrt{a^2-x^2}, \sqrt{a^2-y^2}) \le z \le \min(\sqrt{a^2-x^2}, \sqrt{a^2-y^2})$.

$$\iiint_D dx\,dy\,dz = \iint_{\tilde{D}} \left(\int_{-\min(\sqrt{a^2-x^2}, \sqrt{a^2-y^2})}^{\min(\sqrt{a^2-x^2}, \sqrt{a^2-y^2})} dz \right) dx\,dy =$$

$$= 2 \iint_{\tilde{D}} \min(\sqrt{a^2-x^2}, \sqrt{a^2-y^2})\, dx\,dy = \Bigg/ \; a^2-x^2 \le a^2-y^2 \Leftrightarrow y^2 \le x^2 \Leftrightarrow -\pi/4 \le \varphi \le \pi/4 \text{ eller } \frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}. \; \Bigg/$$



$$= /\text{Av symmetriskäl}/ = 16 \int_0^{\pi/4} \left(\int_0^a \sqrt{a^2-\rho^2\cos^2\varphi}\; \rho\, d\rho \right) d\varphi =$$

$$= 16 \int_0^{\pi/4} \left[-(a^2-\rho^2\cos^2\varphi)^{3/2} \cdot \frac{1}{3\cos^2\varphi} \right]_{\rho=0}^{a} d\varphi =$$

$$= 16 \int_0^{\pi/4} \left(\frac{a^3}{3\cos^2\varphi} - a^3 \cdot \frac{(1-\cos^2\varphi)^{3/2}}{3\cos^2\varphi} \right) d\varphi = \ldots =$$

6.35)

b) forts.

$$= \frac{16a^3}{3}\int_0^{\pi/4} \frac{1}{\cos^2\varphi}\,d\varphi - \frac{16a^3}{3}\int_0^{\pi/4} \frac{\sin^3\varphi}{\cos^2\varphi}\,d\varphi =$$

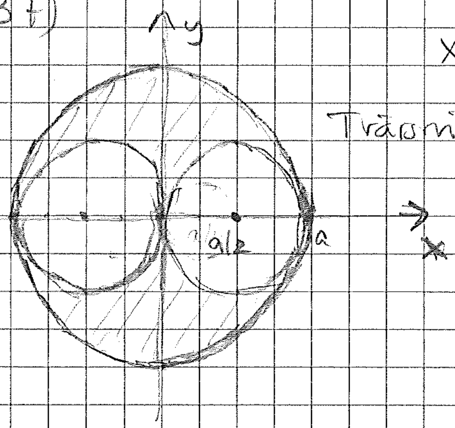
$$= \frac{16a^3}{3}\Big[\tan\varphi\Big]_0^{\pi/4} - \frac{16a^3}{3}\int_0^{\pi/4} \frac{(1-\cos^2\varphi)\sin\varphi}{\cos^2\varphi}\,d\varphi =$$

$$= \frac{16a^3}{3} - \frac{16a^3}{3}\int_0^{\pi/4} \left(\frac{\sin\varphi}{\cos^2\varphi} - \sin\varphi\right)d\varphi =$$

$$= \frac{16a^3}{3} - \frac{16a^3}{3}\left[\frac{1}{\cos\varphi} + \cos\varphi\right]_0^{\pi/4} =$$

$$= \frac{16a^3}{3}\left(1 - \frac{1}{\sqrt{2}} - \sqrt{2} + 1 + 1\right) = \underline{\underline{8a^3(2-\sqrt{2})}}$$

6.37)



$x^2+y^2+z^2 \le a^2$

Tvärsnitt för $z=0$

Hål $\left(x-\frac{a}{2}\right)^2+y^2 \le \frac{a^2}{4}$

$\left(x+\frac{a}{2}\right)^2+y^2 \le \frac{a^2}{4}$

I detta fall är tvärsnittet för $z=0$ också projektionen av D på xy-planet.

Volymen ges nu av Volym(Klot) − Volym hål:

$$\frac{4\pi a^3}{3} - 2\iint_B \left(\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz \right) dx\,dy \quad \Big/ \text{där } B: \left(x-\frac{a}{2}\right)^2+y^2 \le \frac{a^2}{4} \Big/$$

$$= \frac{4\pi a^3}{3} - 4\iint_B \sqrt{a^2-x^2-y^2}\,dx\,dy = \left/ \begin{array}{l} \left(\rho\cos\varphi - \frac{a}{2}\right)^2 + \rho^2\sin^2\varphi \le \frac{a^2}{4} \\ \Leftrightarrow \rho^2 \le a\rho\cos\varphi \Leftrightarrow \\ 0 \le \rho \le a\cos\varphi,\ -\pi/2 \le \varphi \le \pi/2 \end{array} \right/$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2} \left(\int_0^{a\cos\varphi} \sqrt{a^2-\rho^2}\,\rho\,d\rho \right) d\varphi =$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2} \left[-\frac{1}{3}\left(a^2-\rho^2\right)^{3/2} \right]_{\rho=0}^{a\cos\varphi} d\varphi =$$

$$= \frac{4\pi a^3}{3} - 4\int_{-\pi/2}^{\pi/2} \left(\frac{a^3}{3} - \frac{a^3}{3}\left(1-\cos^2\varphi\right)^{3/2} \right) d\varphi =$$

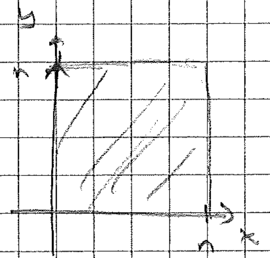
$$= \frac{4\pi a^3}{3} - \frac{4\pi a^3}{3} + \frac{4a^3}{3}\int_{-\pi/2}^{\pi/2} \left(1-\cos^2\varphi\right)^{3/2} d\varphi =$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \left(1-\cos^2\varphi\right)\sin\varphi\,d\varphi = \frac{8a^3}{3}\left[-\cos\varphi + \frac{\cos^3\varphi}{3} \right]_0^{\pi/2} = \underline{\underline{\frac{16a^3}{9}}}$$

6.43)

a)

$D_n = \{(x,y): 0 \le x \le n,\ 0 \le y \le n\}$



$$\iint_{D_n} e^{-x-y}\,dx\,dy = \int_0^n \left(\int_0^n e^{-x-y}\,dy \right) dx =$$

$$= \int_0^n \left[-e^{-x-y} \right]_{y=0}^{n} dx = \int_0^n \left(-e^{-n-x} + e^{-x} \right) dx =$$

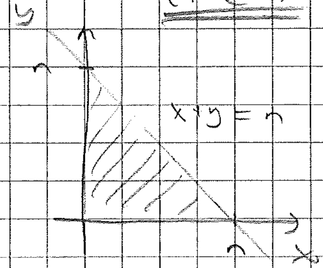
$$= \left[e^{-n-x} - e^{-x} \right]_0^n = e^{-2n} - e^{-n} - e^{-n} + e^0 = 1 + \left(e^{-2n} - 2e^{-n} \right) =$$

$$= \underline{\underline{(1-e^{-n})^2}}.$$

b)

$D_n = \{(x,y): x+y \le n,\ x \ge 0,\ y \ge 0\} =$

$= \{(x,y): 0 \le x \le n,\ 0 \le y \le n-x\}$



$$\iint_{D_n} e^{-x-y}\,dx\,dy = \int_0^n \left(\int_0^{n-x} e^{-x-y}\,dy \right) dx =$$

$$= \int_0^n \left[-e^{-x-y} \right]_{y=0}^{n-x} dx = \int_0^n \left(-e^{-n} + e^{-x} \right) dx =$$

$$= \left[-xe^{-n} - e^{-x} \right]_0^n = -ne^{-n} - e^{-n} + e^0 = \underline{\underline{1 - (1+n)e^{-n}}}$$

c)

$$\iint_D e^{-x-y}\,dx\,dy = \lim_{n\to\infty} \iint_{D_n} e^{-x-y}\,dx\,dy =$$

$$= \lim_{n\to\infty} \left(1 + \left(e^{-2n} - 2e^{-n} \right) \right) = \underline{\underline{1}}$$

(Alt. $\lim_{n\to\infty} \left(1 - (1+n)e^{-n} \right) = 1$).

6.44)

a)

$$\iint_D \frac{xy}{(1+x^2+y^2)^3}\,dx\,dy = \Big/ \begin{array}{l} D=\{(x,y): x\geq 0,\ y\geq 0\} \\ D_n=\{(x,y): x^2+y^2\leq n^2,\ 0\leq\varphi\leq\pi/2\} \\ \text{OBS! Positiv integrand} \end{array} \Big/$$

$$= \lim_{n\to\infty} \int_0^{\pi/2}\left(\int_0^n \frac{\rho^2\cos\varphi\sin\varphi}{(1+\rho^2)^3}\rho\,d\rho\right)d\varphi = \Big/ \begin{array}{l} \rho^2=t \\ 2\rho\,d\rho=dt \\ 0\mapsto 0,\ n\mapsto\sqrt{n} \end{array} \Big/$$

$$= \lim_{n\to\infty} \frac{1}{2}\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi \cdot \int_0^{\sqrt{n}} \frac{t}{(1+t)^3}\,dt =$$

$$= \Big/ \begin{array}{l} \frac{t}{(1+t)^3} = \frac{1}{(1+t)^2} - \frac{1}{(1+t)^3} \\ \cos\varphi\sin\varphi = \frac{\sin 2\varphi}{2} \end{array} \Big/ =$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\frac{-\cos 2\varphi}{4}\right]_0^{\pi/2} \cdot \left[\frac{-1}{(1+t)} + \frac{1}{2(1+t)^2}\right]_0^{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(\frac{-1}{1+\sqrt{n}} + \frac{1}{2(1+\sqrt{n})^2} + 1 - \frac{1}{2}\right) = \underline{\underline{\frac{1}{8}}}$$

b)

$D_n=\{(x,y): x^2+y^2\leq n^2\}$ OBS! Positiv integrand.

$$\iint_{\mathbb{R}^2} x^2 e^{-\sqrt{x^2+y^2}}\,dx\,dy = \lim_{n\to\infty}\int_0^{2\pi}\left(\int_0^n \rho^2\cos^2\varphi\, e^{-\rho}\rho\,d\rho\right)d\varphi =$$

$$= \lim_{n\to\infty}\int_0^{2\pi}\cos^2\varphi\,d\varphi \cdot \int_0^n \rho^3 e^{-\rho}\,d\rho =$$

$$= \Big/ \cos^2\varphi = \frac{\cos 2\varphi+1}{2},\quad \int \rho^3 e^{-\rho}d\rho = -\rho^3e^{-\rho} + \int 3\rho^2 e^{-\rho}d\rho$$

$$= -\rho^3e^{-\rho} - 3\rho^2e^{-\rho} + \int 6\rho e^{-\rho}d\rho = -\rho^3e^{-\rho} - 3\rho^2e^{-\rho} - 6\rho e^{-\rho} - 6e^{-\rho} \Big/$$

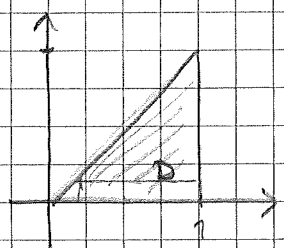
$$= \lim_{n\to\infty} \pi\left[-\rho^3e^{-\rho} - 3\rho^2e^{-\rho} - 6\rho e^{-\rho} - 6e^{-\rho}\right]_0^n =$$

$$= \lim_{n\to\infty} \pi\left[-n^3e^{-n} - 3n^2e^{-n} - 6ne^{-n} - 6e^{-n} + 6\right] = \underline{\underline{6\pi}}$$

6.45)

a) $D = \{(x,y):\ 0 < y \le x \le 1\}$

$= \{(x,y):\ 0 < x \le 1,\ 0 < y \le x\}$



Integrand positiv på D, integral generaliserad i $y=0$.

$D_n = \{(x,y):\ \frac{1}{n} \le x \le 1,\ \frac{1}{n} \le y \le x\}$

$$\iint_D \frac{x}{y}\,dxdy = \lim_{n\to\infty} \int_{1/n}^{1} \left(\int_{1/n}^{x} \frac{x}{y}\,dy \right) dx =$$

$$= \lim_{n\to\infty} \int_{1/n}^{1} \Big[x\ln|y| \Big]_{y=1/n}^{x} dx = \lim_{n\to\infty} \int_{1/n}^{1} (x\ln x + x\ln n)\,dx$$

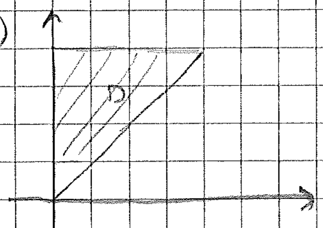
$$= \Big/ \int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} \Big/$$

$$= \lim_{n\to\infty} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{x^2}{2}\ln n \right]_{1/n}^{1} =$$

$$= \lim_{n\to\infty} \left[-\frac{1}{4} + \frac{1}{2}\ln n - \frac{1}{2n^2}\ln n + \frac{1}{4n^2} - \frac{1}{2n^2}\ln n \right] = \infty.$$

D.v.s. <u>DIVERGENT</u>

b)



$D_n = \{(x,y):\ \frac{1}{n} \le x \le y,\ \frac{1}{n} \le y \le 1\}$

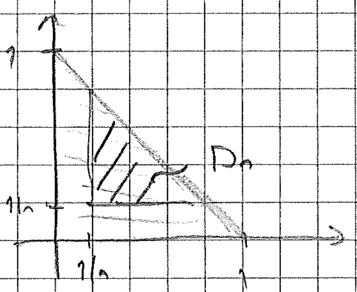
$$\iint_D \frac{x}{y}\,dxdy = \lim_{n\to\infty} \int_{1/n}^{1} \left(\int_{1/n}^{y} \frac{x}{y}\,dx \right) dy =$$

$$= \lim_{n\to\infty} \int_{1/n}^{1} \left[\frac{x^2}{2y} \right]_{x=1/n}^{y} dy = \lim_{n\to\infty} \int_{1/n}^{1} \left(\frac{y}{2} - \frac{1}{2n^2 y} \right) dy =$$

$$= \lim_{n\to\infty} \left[\frac{y^2}{4} - \frac{1}{2n^2}\ln|y| \right]_{1/n}^{1} = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{4n^2} - \frac{1}{2n^2}\ln n \right) = \underline{\underline{\frac{1}{4}}}$$

6.46b)

$$D = \{(x,y) : x+y \le 1,\ x>0, y>0\} = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$$



Positiv integrand. Generaliserad i $x=0$ och $y=0$

$$D_n = \{(x,y) : \frac{1}{n} \le x \le 1,\ \frac{1}{n} \le y \le 1-x\}$$

$$\iint_D \frac{1}{\sqrt{xy}}\,dx\,dy = \lim_{n\to\infty} \iint_{D_n} \frac{1}{\sqrt{xy}}\,dx\,dy =$$

$$= \lim_{n\to\infty} \int_{1/n}^{1} \left(\int_{1/n}^{1-x} \frac{1}{\sqrt{x}\sqrt{y}}\,dy \right) dx = \lim_{n\to\infty} \int_{1/n}^{1} \left[\frac{2\sqrt{y}}{\sqrt{x}} \right]_{y=1/n}^{1-x} dx =$$

$$= \lim_{n\to\infty} \int_{1/n}^{1} \left(\frac{2\sqrt{1-x}}{\sqrt{x}} - \frac{2\sqrt{1/n}}{\sqrt{x}} \right) dx =$$

$$= \lim_{n\to\infty} 2 \left(\int_{1/n}^{1} \frac{\sqrt{1-x}}{\sqrt{x}}\,dx - \left[2\sqrt{1/n}\,\sqrt{x} \right]_{1/n}^{1} \right)$$

$$= \left/ \begin{array}{l} x = t^2 \Leftrightarrow t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \\ 1/n \mapsto \frac{1}{\sqrt{n}},\ 1 \mapsto 1 \end{array} \right/ = \lim_{n\to\infty} 4 \left(\int_{1/\sqrt{n}}^{1} \sqrt{1-t^2}\,dt \underbrace{- 2\sqrt{1/n} + 2\sqrt{1/n}\cdot\sqrt{1/n}}_{\to 0} \right)$$

$$= \left/ \begin{array}{l} t = \sin\theta \\ dt = \cos\theta\,d\theta \\ 1/\sqrt{n} \mapsto \arcsin 1/\sqrt{n} =: \theta_n,\ 1 \mapsto \pi/2 \end{array} \right/ =$$

$$= \lim_{n\to\infty} 4 \left(\int_{\theta_n}^{\pi/2} \cos^2\theta\,d\theta \right)$$

$$= \left/ \cos^2\theta = \frac{\cos 2\theta + 1}{2} \right/ = \lim_{n\to\infty} 4 \left(\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2} \right]_{\theta_n}^{\pi/2} \right)$$

$$= \lim_{n\to\infty} 4 \left(\frac{\sin 2\theta_n}{4} + \frac{\pi}{4} - \frac{\theta_n}{2} \right) = \left/ \theta_n \to 0 \right/ = \underline{\underline{\pi}}$$

6.48)

$$\iint_{R^2} e^{-x^2-xy-y^2}\,dx\,dy =$$

$x^2+xy+y^2 = (x+\frac{y}{2})^2+\frac{3y^2}{4}$

$D_n = \{(x,y) : (x+\frac{y}{2})^2+\frac{3y^2}{4} \le n^2\}$

Positiv integrand.

$$= \lim_{n\to\infty} \iint_{D_n} e^{-(x+\frac{y}{2})^2 - \frac{3y^2}{2}}\,dx\,dy =$$

$$= \quad u = x+\frac{y}{2}, \quad v = \frac{\sqrt{3}\,y}{2}, \quad \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 1/2 \\ 0 & \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{\sqrt{3}}{2}$$

D_n avbildas på $\Omega_n = \{(u,v) : u^2+v^2 \le n^2\}$

$dx\,dy = \frac{2}{\sqrt{3}}\,du\,dv$

$$= \frac{2}{\sqrt{3}} \lim_{n\to\infty} \iint_{\Omega_n} e^{-u^2-v^2}\,du\,dv = \frac{2}{\sqrt{3}} \lim_{n\to\infty} \int_0^{2\pi}\left(\int_0^n e^{-\rho^2}\rho\,d\rho\right)d\varphi$$

$$= \frac{4\pi}{\sqrt{3}} \lim_{n\to\infty} \left[-\frac{e^{-\rho^2}}{2}\right]_0^n = \underline{\underline{\frac{2\pi}{\sqrt{3}}}}$$

6.50)

$$\iiint_{\mathbb{R}^3\setminus\{(0,0,0)\}} \frac{\exp(-\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = / \text{Positiv integrand} / =$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_{1/n}^{n} \frac{e^{-r}}{r} r^2 \sin\theta \, dr \right) d\theta \right) d\varphi =$$

$$= \lim_{n\to\infty} 2\pi \int_0^{\pi} \sin\theta \, d\theta \cdot \int_{1/n}^{n} r e^{-r} \, dr =$$

$$= \lim_{n\to\infty} 2\pi \left[-\cos\theta \right]_0^{\pi} \left[-re^{-r} - e^{-r} \right]_{1/n}^{n} =$$

$$= 4\pi \lim_{n\to\infty} \left(-ne^{-n} - e^{-n} + \frac{1}{n} e^{-1/n} + e^{-1/n} \right) = \underline{\underline{4\pi}}$$

6.51)

$$D=\{(x,y,z):\ \sqrt{x^2+y^2}<z<1\}=$$
$$=\{(x,y,z):\ 0<z<1\ ,\ \sqrt{x^2+y^2}<z\}$$

$$\iiint_D \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \Big/ \begin{array}{l}\text{Positiv integrand} \\ D_n=\{(x,y,z):\ \frac{1}{n}<z\leq 1,\ \sqrt{x^2+y^2}<z\} \\ \Omega_z=\{(x,y):\ \sqrt{x^2+y^2}<z\}\end{array}\Big/ =$$

$$=\lim_{n\to\infty}\iiint_{D_n}\frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz=$$

$$=\lim_{n\to\infty}\int_{1/n}^{1}\left(\iint_{\Omega_z}\frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\right)dz=$$

$$=\lim_{n\to\infty}\int_{1/n}^{1}\left(\int_0^{2\pi}\left(\int_0^{z}\frac{1}{\sqrt{\rho^2+z^2}}\,\rho\,d\rho\right)d\varphi\right)dz=$$

$$=\lim_{n\to\infty}\int_{1/n}^{1}\left(2\pi\left[\sqrt{\rho^2+z^2}\right]_{\rho=0}^{z}\right)dz=$$

$$=2\pi\lim_{n\to\infty}\int_{1/n}^{1}\left(\sqrt{2}\,z-z\right)dz=2\pi\lim_{n\to\infty}\left[\frac{(\sqrt{2}-1)}{2}z^2\right]_{1/n}^{1}=$$

$$=2\pi\lim_{n\to\infty}\left(\frac{\sqrt{2}-1}{2}-\frac{\sqrt{2}-1}{2}\cdot\frac{1}{n^2}\right)=\underline{\underline{(\sqrt{2}-1)\pi}}$$

6.47)

$$D = \{(x,y) : x \geq 0,\ y \geq 0\}$$

$$\iint_D \frac{1}{1+(x+y)^4}\, dx\,dy = \left/ \begin{array}{l} \text{Positiv integrand}, \quad u = x,\ v = x+y \\ \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \\ D \text{ avbildas på } \Omega = \{(u,v) : 0 \leq v < \infty,\ 0 \leq u \leq v\} \end{array} \right/$$

$$= \iint_\Omega \frac{1}{1+v^4}\, du\,dv = \lim_{n\to\infty} \int_0^n \left(\int_0^v \frac{1}{1+v^4}\, du \right) dv =$$

$$= \lim_{n\to\infty} \int_0^n \frac{v}{1+v^4}\, dv = \left/ \begin{array}{l} v^2 = t \\ v\,dv = \frac{dt}{2} \\ 0 \mapsto 0,\ n \mapsto n^2 \end{array} \right/ =$$

$$= \lim_{n\to\infty} \int_0^{n^2} \frac{1}{2(1+t^2)}\, dt = \lim_{n\to\infty} \frac{1}{2}\left(\arctan n^2 - \arctan 0 \right) = \underline{\underline{\frac{\pi}{4}}}$$

6.52)

a) $$\iint_{\mathbb{R}^2} \frac{x}{1+x^2+y^2}\,dxdy = \left/ \begin{array}{l} \text{OBS! Integrand växlar tecken} \\ \Omega_+ = \{(x,y) : x>0\},\ \Omega_- = \{(x,y): x<0\} \end{array} \right/$$

$$= \iint_{\Omega_+} \frac{x}{1+x^2+y^2}\,dxdy + \iint_{\Omega_-} \frac{x}{1+x^2+y^2}\,dxdy$$

Eftersom

$$\iint_{\Omega_+} \frac{x}{1+x^2+y^2}\,dxdy = \lim_{n\to\infty} \int_{-\pi/2}^{\pi/2} \left(\int_0^n \frac{\rho\cos\varphi}{1+\rho^2}\,\rho\,d\rho \right) d\varphi =$$

$$= 2\lim_{n\to\infty} \int_0^n \frac{\rho^2}{1+\rho^2}\,d\rho = 2\lim_{n\to\infty} \left[\rho - \arctan\rho\right]_0^n = \infty$$

Så är hela integralen divergent.

(P.s.s. är $\iint_{\Omega_-} \frac{x}{1+x^2+y^2}\,dxdy = -\infty$, men notera

att konvergens innebär att bägge integralerna

över Ω_+ och Ω_- måste vara ändliga!)

6.52)

b)

Eftersom $\left|\frac{x}{1+(x^2+y^2)^2}\right| \le \frac{\rho}{1+\rho^4}$ och

$$\int_0^{2\pi}\left(\int_0^{\infty}\frac{\rho}{1+\rho^4}\rho\,d\rho\right)d\varphi = 2\pi\int_0^{\infty}\frac{\rho^2}{1+\rho^4}d\rho \le 2\pi\left(\int_0^1 1\,d\rho + \int_1^{\infty}\frac{1}{\rho^2}d\rho\right)$$

$< \infty$ så är integralen absolutkonvergent.

Av symmetriskäl ser vi att

$$\int_{-n}^{n}\left(\int_{-n}^{n}\frac{x}{1+(x^2+y^2)^2}dx\right)dy = 0 \quad \text{för varje } n,$$

så därför är integralens värde $\underline{\underline{0}}$.

c)

$D = \{(x,y) : x \ge 0,\ y \ge 0\}$.

Variabelbytet $\begin{cases} u = x-2y \\ v = 2x+y \end{cases}$ uppfyller $\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$.

Det avbildar D på $\Omega = \{(u,v) : u+2v \ge 0,\ v-2u \ge 0,\ v \ge 0\}$

$$\frac{1}{5}\iint_\Omega u e^{-v}\,du\,dv = \frac{1}{5}\int_0^{\infty}\left(\int_0^{v/2} u e^{-v}\,du\right)dv + \frac{1}{5}\int_0^{\infty}\left(\int_{-2v}^{0} u e^{-v}\,du\right)dv$$

$$= \frac{1}{5}\int_0^{\infty}\left[\frac{u^2}{2}e^{-v}\right]_{u=0}^{v/2}dv + \frac{1}{5}\int_0^{\infty}\left[\frac{u^2}{2}e^{-v}\right]_{u=-2v}^{0}dv =$$

$$= \frac{1}{5}\int_0^{\infty}\frac{v^2}{8}e^{-v}dv + \frac{1}{5}\int_0^{\infty}-2v^2e^{-v}dv = \left/ \int v^2e^{-v}dv = -v^2e^{-v}+2\int ve^{-v}dv = -v^2e^{-v}-2ve^{-v}-2e^{-v} \right/$$

$$= \text{"}\frac{1}{5}\frac{1}{8}\left[-v^2e^{-v}-2ve^{-v}-2e^{-v}\right]_0^{\infty} + \frac{2}{5}\left[-v^2e^{-v}-2ve^{-v}-2e^{-v}\right]_0^{\infty}\text{"} =$$

$$= \frac{2}{40} - \frac{4}{5} = -\frac{15}{20} = \underline{\underline{-\frac{3}{4}}}.$$

Eftersom båda integralerna ovan är konvergenta motiverar detta att den ursprungliga integralen också är det...

6.53)

a) $$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \underline{\underline{\frac{x^2-y^2}{x^2+y^2}}} = f(x,y)$$

b) $$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = \int_0^1 \left[\frac{y}{x^2+y^2} \right]_{y=0}^{1} dx = \int_0^1 \frac{1}{1+x^2}\,dx =$$

$$= \arctan 1 = \underline{\underline{\frac{\pi}{4}}}.$$

c) $$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_0^1 \left(\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx \right) dy$$

$$= \Big/ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \Big/$$

$$= \int_0^1 \left[\frac{-x}{x^2+y^2} \right]_{x=0}^{1} dy = \int_0^1 \frac{-1}{1+y^2}\,dy = -\arctan 1 = -\frac{\pi}{4}.$$

d)

<u>Divergent</u> ty annars skulle vi fått samma värde i b) och c).

Vi kan också se detta eftersom området $0 \le x \le 1$, $0 \le y \le x$ ligger i D, och

$$\int_0^1 \left(\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \right) dx = \int_0^1 \left[\frac{y}{x^2+y^2} \right]_{y=0}^{x} dx =$$

$$= \int_0^1 \frac{x}{2x^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x}\,dx = \infty.$$