2.1 a) $f(x,y) = x + x^3 y + x^2 y^3 + y^5$

$f'_x = 1 + 3x^2 y + 2xy^3$

$f'_y = x^3 + 3x^2 y^2 + 5y^4$

b) $f(x,y) = \ln(1 - x^2 - 2y^2)$

$f'_x = \frac{1}{1-x^2-2y^2} \cdot (-2x) = \frac{-2x}{1-x^2-2y^2}$

$f'_y = \frac{1}{1-x^2-2y^2} (-4y) = \frac{-4y}{1-x^2-2y^2}$

c) $f(x,y) = e^{-y^2} \arcsin 2y$

$f'_x = 0$

$f'_y = -2y \cdot e^{-y^2} \arcsin 2y + e^{-y^2} \frac{2}{\sqrt{1-(2y)^2}}$

$= e^{-y^2} \left(\frac{2}{\sqrt{1-4y^2}} - 2y \arcsin 2y \right)$

d) $f(x,y) = \frac{x+y}{x-y}$

$f'_x = \frac{1 \cdot (x-y) - 1 \cdot (x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$f'_y = \frac{1 \cdot (x-y) + 1 (x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

2.2 a)

$$f(x,y,z) = \cos(xy - z^2)$$

$$f'_x = -\sin(xy - z^2) \cdot y = -y\sin(xy - z^2)$$

$$f'_y = -x\sin(xy - z^2)$$

$$f'_z = (-2z) \cdot (-\sin(xy - z^2)) = 2z\sin(xy - z^2)$$

b)

$$f(x,y,z) = \frac{1}{\sqrt{z}} \arctan\frac{y}{x}$$

$$f'_x = \frac{1}{\sqrt{z}} \frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{\sqrt{z}(x^2+y^2)}$$

$$f'_y = \frac{1}{\sqrt{z}} \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{\sqrt{z}(x^2+y^2)}$$

$$f'_z = -\frac{1}{2z\sqrt{z}} \arctan\frac{y}{x}.$$

2.3.

$$f''_{xx} = (f'_x)'_x = \frac{\partial}{\partial x}(1 + 3x^2y + 2xy^3 + 5y^4) = 6xy + 2y^3$$

$$f''_{xy} = (f'_x)'_y = \frac{\partial}{\partial y}\left(1 + 3x^2y + 2xy^3\right) = 3x^2 + 6xy^2$$

$$f''_{yx} = (f'_y)'_x = \frac{\partial}{\partial x}(x^3 + 3x^2y^2 + 5y^4) = 3x^2 + 6xy^2$$

$$f''_{yy} = (f'_y)'_y = \frac{\partial}{\partial y}(x^3 + 3x^2y^2 + 5y^4) = 6x^2y + 20y^3$$

2.7)

a) $\begin{cases} z'_x = 2x+y \\ z'_y = x+2y \end{cases}$

$z'_x = 2x+y \Rightarrow z = x^2 + xy + g(y) \Rightarrow$

$\Rightarrow z'_y = x + g'(y) = x+2y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C$

SVAR: $z = x^2 + xy + y^2 + C.$

b) $\begin{cases} z'_x = e^{xy} \\ z'_y = e^{xy} \end{cases}$

$z'_x = e^{xy} \Rightarrow z = \frac{1}{y} e^{xy} + g(y) \Rightarrow z'_y = -\frac{1}{y^2} e^{xy} + \frac{x}{y} e^{xy} + g'(y)$

$= e^{xy} \Leftrightarrow g'(y) = e^{xy}\left(1 + \frac{1}{y^2} - \frac{x}{y}\right)$

Går ej, så lösning saknas.

c) $\begin{cases} z'_x = ye^x \\ z'_y = 1+e^x \end{cases}$

$z'_x = ye^x \Rightarrow z = ye^x + g(y) \Rightarrow z'_y = e^x + g'(y) = 1 + e^x$

$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y + C.$

SVAR: $z = ye^x + y + C.$

2.9)

a) $\begin{cases} u'_x = y+3z-3 \\ u'_y = x+2z-2 \\ u'_z = 2y+3x-1 \end{cases}$

$u'_x = y+3z-3 \Rightarrow u = xy+3xz-3x+h(y,z) \Rightarrow$

$\Rightarrow u'_y = x + h'_y = x+2z-2 \Rightarrow h'_y = 2z-2$

$\Rightarrow h = 2yz-2y+g(z) \Rightarrow u = xy+3xz-3x+2yz-2y+g(z)$

$\Rightarrow u'_z = 3x+2y+g'(z) = 2y+3x-1$

$\Rightarrow g'(z) = -1 \Rightarrow g(z) = -z+C$

SVAR: $u = xy+3xz-3x+2yz-2y-z+C$

b) $\begin{cases} u'_x = 1+y\sin xy \\ u'_y = e^z + x\sin xy \\ u'_z = (1+y+z)e^z. \end{cases}$

$u'_x = 1+y\sin xy \Rightarrow u = x - \cos xy + h(y,z)$

$\Rightarrow u'_y = x\sin xy + h'_y = e^z + x\sin xy \Rightarrow h'_y = e^z$

$\Rightarrow h = ye^z + g(z) \Rightarrow u = x-\cos xy + ye^z + g(z)$

$\Rightarrow u'_z = ye^z + g'(z) = (1+y+z)e^z$

$\Rightarrow g'(z) = (1+z)e^z \Rightarrow g(z) = ze^z + C$

SVAR: $u = x-\cos xy + ye^z + ze^z + C.$

c) $\begin{cases} u'_x = z+xy^2 \\ u'_y = x^2y \\ u'_z = yz \end{cases}$

$u'_x = z+xy^2 \Rightarrow u = xz + \frac{x^2y^2}{2} + h(y,z)$

$\Rightarrow u'_y = x^2y + h'_y = x^2y \Rightarrow h'_y = 0$

$\Rightarrow h = g(z) \Rightarrow u = xz + \frac{x^2y^2}{2} + g(z)$

$\Rightarrow u'_z = x + g'(z) = yz$

Går ej, så lösning saknas.

2.11 a) $z = z(x,y)$

$\frac{\partial z}{\partial x} = 0 \iff z = h(y) \quad h \in C^2(\mathbb{R})$.

b) $\frac{\partial z}{\partial y} = 0 \iff z = h(x) \quad h \in C^2(\mathbb{R})$.

c) $\frac{\partial^2 z}{\partial x^2} = 0 \Rightarrow \frac{\partial z}{\partial x} = h(y) \Rightarrow z = x h(y) + g(y) \quad h, g \in C^2(\mathbb{R})$

d) $\frac{\partial^2 z}{\partial x \partial y} = 0 \Rightarrow \frac{\partial z}{\partial x} = h(x) \Rightarrow z = H(x) + K(y) \quad H, K \in C^2(\mathbb{R})$.

e) $\frac{\partial z}{\partial x} = z \iff e^{-x}\frac{\partial z}{\partial x} = e^{-x} z \iff \frac{\partial}{\partial x}(e^{-x} z) = 0$

$\iff e^{-x} z = h(y) \qquad \underline{\underline{z = h(y) e^{x} \quad h \in C^2(\mathbb{R})}}$

f) $\frac{\partial z}{\partial x} = yz \iff e^{-xy}\frac{\partial z}{\partial x} = e^{-xy} y z$

$\iff \frac{\partial}{\partial x}(e^{-xy} z) = 0 \iff e^{-xy} z = h(y)$

$\underline{\underline{z = h(y) e^{xy} \quad h \in C^2(\mathbb{R})}}$

OBS! I (e), (f) används metod med integrerande faktor. JMF med envariabel2

g) $\frac{\partial z}{\partial x} = xz \iff e^{-x^2/2}\frac{\partial z}{\partial x} = e^{-x^2/2} z \iff$

$\iff \frac{\partial}{\partial x}(e^{-x^2/2} z) = 0 \iff e^{-x^2/2} z = h(y)$

$\underline{\underline{z = h(y) e^{x^2/2} \quad h \in C^2(\mathbb{R})}}$.

h) $\frac{\partial^2 z}{\partial y^2} + e^{2x} z = 0$. Behandlar vi x som konstant, då vi bara har derivator m.a.p. y blir detta andra ordningens ekv. med konst. koeff.

$r^2 + e^{2x} = 0 \iff r = \pm i e^{x}$.

$\underline{\underline{z = h(x)\cos(ye^{x}) + g(x)\sin(ye^{x}) \quad h, g \in C^2(\mathbb{R})}}$.

2.12)

a)

$$z(x,y) = x^3 + xy^2$$

$z(1+h, 2+k) = z(1,2) + z'_x(1,2)h + z'_y(1,2)k +$ restterm.

$z(1,2) = 1 + 1 \cdot 2^2 = 5$, så $(1,2,5)$ ligger på grafen

$z'_x = 3x^2 + y^2$, $z'_x(1,2) = 3 + 4 = 7$

$z'_y = 2xy$, $z'_y(1,2) = 2 \cdot 1 \cdot 2 = 4$

Tangentplan ges av

$$z = 5 + 7h + 4k = 5 + 7(x-1) + 4(y-2)$$

b)

$z(x,y) = e^{2x} - 1$. $z(0,2) = 1 - 1 = 0$, så $(0,2,0)$ ligger på grafen.

$z'_x = 2e^{2x}$, $z'_x(0,2) = 2$

$z'_y = 0$.

$$z = 0 + 2(x-0) + 0(y-2) = 2x.$$

c)

$y(x,z) = \arcsin xz$, $y(1, 1/2) = \arcsin 1/2 = \pi/6$ ok

$y'_x = \dfrac{1}{\sqrt{1-x^2z^2}} \cdot z$, $y'_x(1, 1/2) = \dfrac{1/2}{\sqrt{1-1/4}} = \dfrac{1}{\sqrt{3}}$

$y'_z = \dfrac{1}{\sqrt{1-x^2z^2}} x$, $y'_z(1, 1/2) = \dfrac{1}{\sqrt{1-1/4}} = \dfrac{2}{\sqrt{3}}$

$$y = \frac{\pi}{6} + \frac{1}{\sqrt{3}}(x-1) + \frac{2}{\sqrt{3}}\left(z - \frac{1}{2}\right) =$$

$$= \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}z + \frac{\pi}{6} - \frac{2}{\sqrt{3}}$$

2.13 a) $f(x,y,z) = 2x - 3y + z$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz =$$

$$= 2\,dx - 3\,dy + dz$$

b) $f(x,y) = \sin xy^2$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y^2\cos xy^2\,dx + 2xy\cos xy^2\,dy$$

c)

$$f(p,V,T) = \frac{pV}{T}$$

$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial V}dV + \frac{\partial f}{\partial T}dT =$$

$$= \frac{V}{T}dp + \frac{p}{T}dV - \frac{pV}{T^2}dT$$

2.14)

$$P(U,R) = \frac{U^2}{R}$$

$$P(10,2) = 50$$

$$P'_U = \frac{2U}{R}, \quad P'_U(10,2) = 10$$

$$P'_R = -\frac{U^2}{R^2}, \quad P'_R(10,2) = -25$$

$$P(10+h, 2+k) \approx P(10,2) + P'_U(10,2)h + P'_R(10,2)k$$

$$= 50 + 10h - 25k$$

om h, k små. Så

a) $P(10.3, 2.1) - P(10,2) \approx 10 \cdot 0.3 - 25 \cdot 0.1 = \underline{\underline{0.5}}$

b) $P(10.3, 2.2) - P(10,2) \approx 10 \cdot 0.3 - 25 \cdot 0.2 = \underline{\underline{-2}}$

2.4)

$$f(x,y) = \begin{cases} \dfrac{x^3+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Om $(x,y) \neq (0,0)$ gäller

$$f'_x = \frac{3x^2(x^2+y^2) - (x^3+y^4)\cdot 2x}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2 - 2xy^4}{(x^2+y^2)^2}.$$

$$f'_y = \frac{4y^3(x^2+y^2) - (x^3+y^4)\cdot 2y}{(x^2+y^2)^2} = \frac{2y^5 + 4x^2y^3 - 2x^3y}{(x^2+y^2)^2}.$$

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3+0^4}{h^2+0^2} - 0}{h} =$$

$$= \lim_{h\to 0} \frac{h^3}{h^3} = \underline{1}.$$

$$f'_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0^3+k^4}{0^2+k^2} - 0}{k} =$$

$$= \lim_{k\to 0} \frac{k^4}{k^3} = \underline{0}.$$

2.5) $f(x,0) = \dfrac{x^2}{x^2} = 1$, så $f'_x(0,0) = 0$

$f(0,y) = \dfrac{y^2}{y^2} = 1$, så $f'_y(0,0) = 0$.

$f(x,-x) = \dfrac{(x-x)^2}{x^2+(-x)^2} = 0$ $\quad x \neq 0$, så

$\lim_{(x,y)\to(0,0)} f(x,y)$ existerar ej.

(Att f'_x och f'_y existerar för $(x,y) \neq (0,0)$ är klart

då $f(x,y) = \dfrac{(x+y)^2}{x^2+y^2}$ där ...)

2.6) $f(x,y) = \begin{cases} y^2 \arctan \frac{x}{y} & y \neq 0 \\ 0 & y = 0 . \end{cases}$

a) $f'_x = y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y^3}{x^2+y^2} \to 0$ då $(x,y) \to (a,0)$

$f'_y = 2y \arctan\left(\frac{x}{y}\right) + y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2} \left(\frac{-x}{y^2}\right) =$

$= 2y \arctan\left(\frac{x}{y}\right) - \frac{xy^2}{x^2+y^2} \to 0$ då $(x,y) \to (a,0)$

$f'_x(a,0) = \lim_{h\to 0} \frac{f(a+h,0) - f(a,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$

$f'_y(a,0) = \lim_{k \to 0} \frac{f(a,k) - f(a,0)}{k} = \lim_{k\to 0} \frac{k^2 \arctan(a/k) - 0}{k} = 0$

Så f'_x, f'_y ä kontinuerliga i $(a,0)$ för varje a.
Att de är det i området $y \neq 0$ ä klart.

b) $f''_{xy}(0,0) = \lim_{k\to 0} \frac{f'_x(0,k) - f'_x(0,0)}{k} =$

$= \lim_{k \to 0} \frac{\frac{k^3}{k^2} - 0}{k} = 1.$

$f''_{yx}(0,0) = \lim_{h \to 0} \frac{f'_y(h,0) - f'_y(0,0)}{h} =$

$= \lim_{h\to 0} \frac{0-0}{h} = 0$

Eftersom $f''_{xy}(0,0) \neq f''_{yx}(0,0)$ ä f inte C^2.

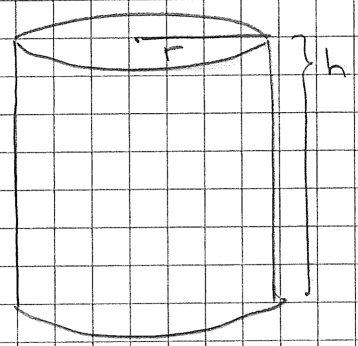
2.8)
$$\begin{cases} z'_x = y e^{x^2y^4} \\ z'_y = x e^{x^2y^4} \end{cases}$$

$$z''_{xy} = (z'_x)'_y = \frac{\partial}{\partial y}(y e^{x^2y^4}) = e^{x^2y^4} + 4y^4x^2e^{x^2y^4}$$

$$z''_{yx} = (z'_y)'_x = \frac{\partial}{\partial x}(x e^{x^2y^4}) = e^{x^2y^4} + 2x^2y^4e^{x^2y^4}.$$

$z''_{xy} \neq z''_{yx}$ så kan ej finnas $z \in C^2$ som uppfyller detta.

2.16)



$$V = \pi r^2 h$$
$$V'_r = 2\pi r h$$
$$V'_h = \pi r^2$$

Om $r-r_0$ och $h-h_0$ små:

$$V(r,h) \approx V(r_0,h_0) + V'_r(r_0,h_0)(r-r_0) + V'_h(r_0,h_0)(h-h_0)$$

$r-r_0 \approx 0.03r_0 \quad h-h_0 \approx -0.01h_0$ ger

$$V(r,h) \approx \pi r_0^2 h_0 + 2\pi r_0 h_0 \cdot 0.03 r_0 - \pi r_0^2 \cdot 0.01 h_0$$
$$= \pi r_0^2 h_0 + \pi r_0^2 h_0 (0.06 - 0.01) = 1.05\, \pi r_0^2 h_0$$

SVAR: ÖKAR MED CA 5%.

2.17)

$f(x,y) = xy$

$f(1+h, 2+k) - f(1,2) =$

$= (1+h)(2+k) - 2 = 2h + k + hk$

$= 2h + k + \frac{hk}{\sqrt{h^2+k^2}} \sqrt{h^2+k^2}$.

$\frac{hk}{\sqrt{h^2+k^2}} \to 0$ då $(h,k) \to (0,0)$

D.v.s. $df(1,2)(h,k) = 2h + k$.

2.18)

(a) Vi noterar att

$$\frac{f(h,k) - f(0,0) - f'_x(0,0)h - f'_y(0,0)k}{\sqrt{h^2+k^2}} =$$

$$= \frac{\frac{h^3+k^4}{h^2+k^2} - h}{\sqrt{h^2+k^2}} = \frac{k^4 - hk^2}{\sqrt{h^2+k^2}(h^2+k^2)}$$

Om f vore differentierbar skulle detta $\to 0$ då $(h,k) \to (0,0)$, men med $h = k$ får vi

$$\frac{k^4 - k^3}{(2k^2)^{3/2}} \to \frac{-1}{2^{3/2}} \text{ då } k \to 0.$$

(b) $f'_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$. P.s.s. $f'_y(0,0) = 0$.

$|f(x,y) - f(0,0)| \leq |y^2 \arctan \frac{x}{y^2}| \leq \frac{\pi}{2} y^2 \leq \frac{\pi}{2}(x^2+y^2)$

så $df(0,0) = 0$.

Men för $y \neq 0$ har vi $f'_x = y^2 \cdot \frac{1}{1 + (\frac{x}{y^2})^2} \cdot \frac{1}{y^2} = \frac{y^4}{x^2+y^4}$

så $f'_x(0,y) = 1 \not\to 0$ då $y \to 0$.