2.42

a) $f(x,y,z) = x + 2y + 3z$

$\operatorname{grad} f = \nabla f = (f'_x, f'_y, f'_z) = (1, 2, 3)$

b)

$f(x,y) = xy^2 e^{-xy}$

$\operatorname{grad} f = \nabla f = (f'_x, f'_y) = (y^2 e^{-xy} - xy^3 e^{-xy}, 2xy e^{-xy} - x^2y^2 e^{-xy}) =$

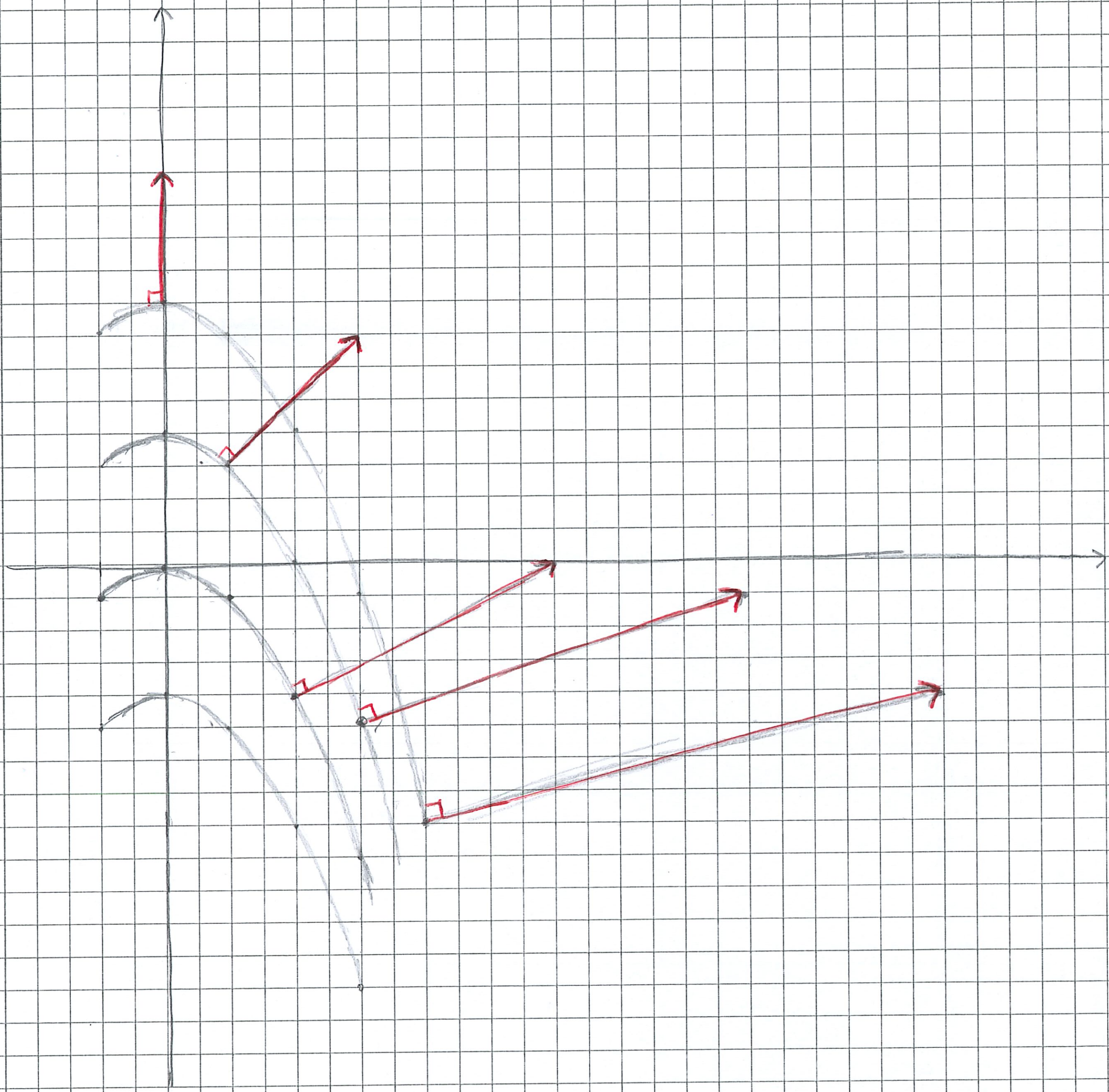
$= ((y^2 - xy^3) e^{-xy}, (2xy - x^2y^2) e^{-xy}).$

2.43) $f(x,y) = x^2 + 4y = C \iff y = \frac{C}{4} - \frac{x^2}{4}$.

$\nabla f = (2x, 4)$ $\quad \nabla f(0,8) = (0,4)$

$\nabla f(2,3) = (4,4)$ $\quad \nabla f(4,-4) = (8,4)$

$\nabla f(6,-5) = (12,4)$ $\quad \nabla f(8,-8) = (16,4)$.



Längre gradient $\Rightarrow$ tätare nivåkurvor.

2.44) $f(x,y) = x^3 + xy + y^3 = 5$

$\nabla f = (f'_x, f'_y) = (3x^2+y,\ x+3y^2)$

$\nabla f(2,-1) = (11,5)$.

(OBS! $f(2,-1) = 5$, så $(2,-1)$ ligger på kurvan).

∇f pekar i normalriktning till nivåkurvan så

$(11,5)\cdot(x-2,\ y+1) = 0 \iff \underline{\underline{11x + 5y = 17}}$ ger tangentlinje.

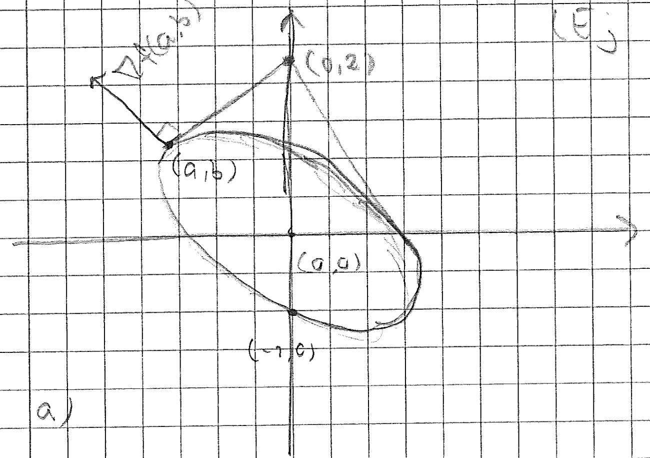
$(5,-11)\cdot(11,5) = 0$ så

$(5,-11)\cdot(x-2,\ y+1) = 0 \iff \underline{\underline{5x - 11y = 21}}$ ger normallinje.

2.45) $f(x,y) = x^2 + xy + y^2 = 1$.

$\nabla f = (2x+y,\ x+2y)$.

(Ej skalenlig bild!)



a)

Vi vill hitta de punkter (a,b) på ellipsen, d.v.s sådana att $a^2+ab+b^2=1$ och sådana att $(a,b)-(0,2) = (a,b-2)$ tangerar ellipsen,

D.v.s. sådana $\nabla f(a,b)\cdot(a,b-2) = 0$.

$$\begin{cases} a^2+ab+b^2=1 \\ \nabla f(a,b)\cdot(a,b-2) = 0 \end{cases} \iff \begin{cases} a^2+ab+b^2=1 \\ (2a+b,\ a+2b)\cdot(a,b-2) = 0 \end{cases}$$

$$\iff \begin{cases} a^2+ab+b^2=1 \\ 2a^2+ab+ab-2a+2b^2-4b=0 \end{cases} \iff \dots \iff \begin{cases} a=1,\ b=0 \\ \text{eller} \\ a=-1,\ b=1. \end{cases}$$

Tangentlinjerna ges nu av

$(1,0)$: $\nabla f(1,0)\cdot(x-1,\ y-0) = (2,1)\cdot(x-1,\ y) = \underline{\underline{2x+y-2=0}}$

$(-1,1)$: $\nabla f(-1,1)\cdot(x+1,\ y-1) = (-1,1)\cdot(x+1,\ y-1) = \underline{\underline{-x+y-2=0}}$

2.45)

b) Vi ser direkt från figuren att det inte finns sådana

c) $(-1,0)$ ligger på ellipsen

$$\nabla f(-1,0)\cdot(x+1,y) = (-2,-1)\cdot(x+1,y) = -2x-y-2 = 0$$

$$\Leftrightarrow \underline{\underline{2x+y=-2}}.$$

2.46)

$$\begin{cases} x^2-y^2=3 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x^2-(2/x)^2=3 \\ y=2/x \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-4/x^2=3 \\ y=2/x \end{cases} \Leftrightarrow \begin{cases} x^4-4-3x^2=0 \\ y=2/x \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} x=2,\ y=1 \\ \text{eller} \\ x=-2,\ y=-1 \end{cases}.$$

Med $f(x,y)=x^2-y^2$, $g(x,y)=xy$

får vi

$$\nabla f = (2x,-2y),\quad \nabla g = (y,x).$$

$$\nabla f(2,1) = (4,-2),\quad \nabla g(2,1) = (1,2)$$

$$\cos\theta = \frac{\nabla f(2,1)\cdot\nabla g(2,1)}{|\nabla f(2,1)|\,|\nabla g(2,1)|} = 0,\quad \text{d.v.s. } \underline{\underline{\theta=\frac{\pi}{2}}}.$$

$$\nabla f(-2,-1) = (-4,2),\quad \nabla g(-2,-1) = (-1,-2)$$

$$\cos\theta = \frac{(-4,2)\cdot(-1,-2)}{|(-4,2)|\,|(-1,-2)|} = 0 \qquad \underline{\underline{\theta=\frac{\pi}{2}}}.$$

2.49)

a)

$f(x,y,z) = x^2 + 2y^2 + 3z^2 = 20 \quad (f(3,-2,-1) = 20 \text{ ok.})$

$\nabla f = (2x, 4y, 6z)$

$\nabla f(3,-2,-1) = (6,-8,-6) = 2(3,-4,-3).$

$(3,-4,-3)\cdot(x-3, y+2, z+1) = \underline{\underline{3x-4y-3z-20=0}}$

b)

$f(x,y,z) = x^2 y - z = 0 \quad (f(-2,1,4) = 0 \text{ ok.})$

$\nabla f = (2xy, x^2, -1)$

$\nabla f(-2,1,4) = (-4,4,-1)$

$(-4,4,-1)\cdot(x+2, y-1, z-4) = \underline{\underline{-4x+4y-z-8=0}}$

2.54)

a) $\nabla f = (f'_x, f'_y)$ pekar i den riktning i vilken f växer snabbast.

b) $\nabla F = (f'_x, f'_y, -1)$ pekar i normalriktningen.

(OBS! Om $F(a,b,c) = 0 \Leftrightarrow c = f(a,b)$

och i använder den linjära approximationen av

f: $z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$

$= c + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$

$\Leftrightarrow (f'_x(a,b), f'_y(a,b), -1)\cdot(x-a, y-b, z-c) =$

$= \nabla F(a,b,c)\cdot(x-a, y-b, z-c) = 0$.)

2.55) a) $f(x,y) = \ln(x^2+2y^2)$

$\nabla f = \left(\frac{2x}{x^2+2y^2}, \frac{4y}{x^2+2y^2}\right)$, $\nabla f(2,1) = \left(\frac{4}{4+2}, \frac{4}{4+2}\right) = \frac{2}{3}(1,1)$.

OBS! $\left|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right| = 1$.

Med $\bar{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ får vi

$f'_{\bar{v}}(2,1) = \nabla f(2,1) \cdot \bar{v} = \frac{2}{3}(1,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \underline{\underline{\frac{2\sqrt{2}}{3}}}$

b) Med $\bar{v} = \frac{(1,2)}{|(1,2)|} = \frac{1}{\sqrt{5}}(1,2)$ får vi

$f'_{\bar{v}}(2,1) = \nabla f(2,1) \cdot \frac{1}{\sqrt{5}}(1,2) = \frac{2}{3}(1,1) \cdot \frac{1}{\sqrt{5}}(1,2) = \underline{\underline{\frac{2}{\sqrt{5}}}}$

2.56) $f(x,y,z) = xy^2z^3$.

$\nabla f = (y^2z^3, 2xyz^3, 3xy^2z^2)$

$\nabla f(3,2,1) = (4, 12, 36)$.

Riktning $\bar{v} = \frac{-(3,2,1)}{|(3,2,1)|} = -\frac{1}{\sqrt{14}}(3,2,1)$

$f'_{\bar{v}}(3,2,1) = \nabla f(3,2,1) \cdot \bar{v} = (4,12,36) \cdot \left(\frac{-1}{\sqrt{14}}(3,2,1)\right) =$

$= \underline{\underline{\frac{-72}{\sqrt{14}}}}$.

2.57) Om vi kallar punkten (a,b) är påståendet

att $\frac{\partial T}{\partial y}(a,b) = -3$, $\frac{\partial T}{\partial x}(a,b) = 2$.

$\nabla T(a,b) = (2,-3)$

a) $(2,-3) \cdot (-1,0) = \underline{\underline{-2}}$

b) $(2,-3) \cdot \frac{(1,-1)}{\sqrt{2}} = \underline{\underline{\frac{5}{\sqrt{2}}}}$.

2.58)

$$T(x,y) = 3\arctan(x^2+y) - 10 - \frac{6}{1+x^2+y^2}.$$

$$\nabla T = \left(\frac{6x}{1+(x^2+y)^2} + \frac{12x}{(1+x^2+y^2)^2}, \frac{3}{1+(x^2+y)^2} + \frac{12y}{(1+x^2+y^2)^2}\right).$$

$$\nabla T(1,-2) = \ldots = \frac{5}{6}(4,1)$$

Riktning $\quad \bar{v} = -\frac{(4,1)}{|(4,1)|} = -\frac{1}{\sqrt{17}}(4,1)$ (Avtar snabbast i $-\nabla T$-riktning).

$$T'_{\bar{v}}(1,-2) = \nabla T(1,-2)\cdot\bar{v} = \frac{5}{6}(4,1)\cdot\left(-\frac{1}{\sqrt{17}}(4,1)\right) =$$

$$= \frac{-5\cdot 17}{6\sqrt{17}} = \frac{-5\sqrt{17}}{6}. \quad (= -|\nabla T|).$$

$$-5\sqrt{17}/6 \approx -3.4\,^\circ C/km$$

$$3\ m/s = \frac{3\cdot 60}{1000}\ km/min$$

$$\frac{3\cdot 60\cdot 5\sqrt{17}}{6\cdot 1000} = \frac{3\sqrt{17}}{20} \approx 0.62\,^\circ C/min.$$

2.48)

$f(x,y) = xy^2 = 2$.

$\nabla f = (y^2, 2xy)$

$(x,y) \parallel \nabla f(x,y) \iff (y,-x) \cdot \nabla f(x,y) = 0$.

$$\begin{cases} (y,-x)\cdot(y^2, 2xy) = 0 \\ xy^2 = 2 \end{cases} \iff \begin{cases} y^3 - 2x^2y = 0 \\ xy^2 = 2 \end{cases} \iff \ldots$$

$\iff \underline{\underline{(x,y) = (1, \pm\sqrt{2})}}$

2.50)

$f(x,y,z) = x^2 + 2y^2 + 3z^2 + 2xy + 2yz = 1$

$\nabla f = (2x+2y,\ 4y+2x+2z,\ 6z+2y)$

V.h. ha $\nabla f \parallel (1,-1,2)$

$$\begin{cases} 2x+2y = \lambda \\ 4y+2x+2z = -\lambda \\ 6z+2y = 2\lambda \\ f(x,y,z) = 1 \end{cases} \Rightarrow \ldots \Rightarrow \underline{\underline{(x,y,z) = \pm\left(\frac{5}{\sqrt{13}}, \frac{-4}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)}}$$

2.51) $f(x,y,z) = x^2 + 2y^2 + 3z^2 = 6$

$\nabla f = (2x, 4y, 6z)$

$$\begin{cases} ((6,0,0) - (x,y,z))\cdot(2x, 4y, 6z) = 0 \\ ((0,3,0) - (x,y,z))\cdot(2x, 4y, 6z) = 0 \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases}$$

$$\iff \begin{cases} 2x^2 + 4y^2 + 6z^2 = 12x \\ 2x^2 + 4y^2 + 6z^2 = 12y \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases} \iff \begin{cases} 12x = 12 \\ 12y = 12 \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases}$$

$\iff (x,y,z) = (1,1,\pm 1)$.

$\nabla f(1,1,1) = (2,4,6) = 2(1,2,3)$ $\qquad (1,2,3)\cdot(x-1, y-1, z-1) = \underline{\underline{x+2y+3z-6=0}}$

$\nabla f(1,1,-1) = 2(1,2,-3)$ $\qquad (1,2,-3)\cdot(x-1, y-1, z+1) = \underline{\underline{x+2y-3z-6=0}}$

2.52) Normal till planet

$((5,-1,1)-(0,1,2)) \times ((1,3,0)-(0,1,2)) =$

$= (5,-2,-1) \times (1,2,-2) = (6,9,12) = 3(2,3,4)$,

så $(2,3,4)$ normal till planet. Ekv. $(2,3,4)\cdot((x,y,z)-(0,1,2)) = 0$

$\Leftrightarrow 2x+3y+4z = 11$

$f(x,y,z) = x^2+3y^2+4z^2 = C$

$\nabla f = (2x, 6y, 8z) = 2(x, 3y, 4z).$

$\nabla f \parallel (2,3,4)$ och $f = C$ ger

$$\begin{cases} (x,3y,4z) = \lambda(2,3,4) \\ x^2+3y^2+4z^2 = C \\ 2x+3y+4z = 11 \end{cases} \Leftrightarrow \begin{cases} x = 2\lambda \\ y = \lambda \\ z = \lambda \\ 4\lambda^2+3\lambda^2+4\lambda^2 = 11\lambda^2 = C \\ 4\lambda+3\lambda+4\lambda = 11 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 2 \\ y = 1 \\ z = 1 \\ C = 11 \\ \lambda = 1 \end{cases}$$

SVAR: $C = 11$.

2.59) $f(x,y) = x+2y-(x-1)^3$. $P = (1,-1)$.

a) $\nabla f = (1-3(x-1)^2, 2)$

$\nabla f(1,-1) = (1,2)$

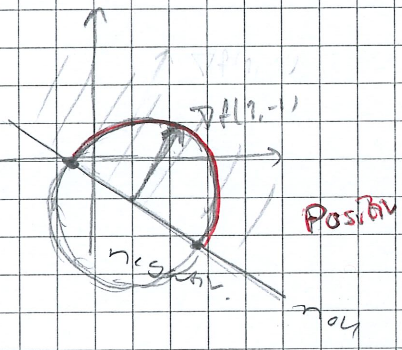
$\nabla f(1,-1)\cdot(a,b) =$

$= a+2b.$

positiv om $a+2b>0$

noll om $a+2b=0$

negativ om $a+2b<0$.



b) Om $a+2b>0 \Rightarrow$ växande initialt.

$a+2b<0 \Rightarrow$ avtagande initialt

Längs $a+2b=0 \Leftrightarrow a=-2b$

$f(-2b,b) = -2b+2b-(-2b-1)^3 = +(2b+1)^3$

Växande initialt om $2b>0$.

2.60)

$$\nabla z = \left(-\frac{8x}{(1+x^2+y^2)^2}, -\frac{8y}{(1+x^2+y^2)^2} \right)$$

$$|\nabla z| = \frac{1}{(1+x^2+y^2)^2} \sqrt{64x^2+64y^2} = \frac{8\sqrt{x^2+y^2}}{(1+x^2+y^2)^2} = \frac{8\rho}{(1+\rho^2)^2} = f(\rho)$$

Maximera $f(\rho)$ $\quad f(0) = 0$

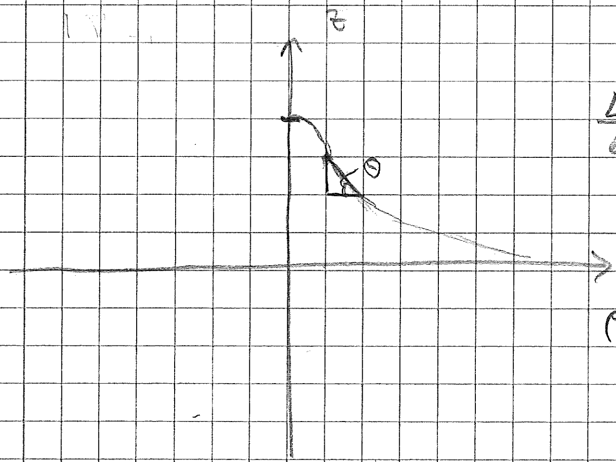
$$f'(\rho) = \frac{8(1+\rho^2)^2 - 8\rho \cdot 2(1+\rho^2)2\rho}{(1+\rho^2)^4} = \frac{8(1+\rho^2-4\rho^2)(1+\rho^2)}{(1+\rho^2)^4} = 0$$

$$\Leftrightarrow \quad 1-3\rho^2 = 0 \quad \Leftrightarrow \quad \rho = 1/\sqrt{3}$$

Vi ser att detta måste vara globalt max då

$f(\rho) \to 0$ då $\rho \to \infty$

D.v.s. max antas på $x^2+y^2 = 1/3$, $z = \dfrac{4}{1+\frac{1}{3}} = 3$.



$$\frac{\Delta z}{\Delta \rho} = \tan\theta$$

$$\theta = \arctan f'(\rho) = \arctan \frac{8 \cdot 1/\sqrt{3}}{(1+1/3)^2} =$$

$$= \arctan \frac{8 \cdot 9}{16\sqrt{3}} =$$

$$= \arctan \frac{3\sqrt{3}}{2} \approx 69^\circ.$$