

2.62)

a) $f(x,y) = 1 - |x| - y^2 \leq 1$, $f(0,0) = 1 \Rightarrow$ lok. (strängt) max

b) $f(x,y) = |x| - \cos y$, $f(0,0) = |0| - \cos 0 = -1$
 $|x| - \cos y \geq -1$, så lokalt min.

c) $f(x,y) = |x| + \cos y$, $f(x,0) > f(0,0)$, men $f(0,y) < f(0,0)$
för $x,y >$ nära 0 så ej extrempunkt

d) $f(x,y,z) = x^2 - yz$, $f(0,0,0) = 0$, $f(x,0,0) = x^2 > 0$, $x \neq 0$
 $f(0,t,t) = -t^2 < 0$ så ej extrempunkt
 $t \neq 0$

e) $f(x,y,z) = \cos x y z$, $f(0,0,0) = 1 \geq \cos x y z$
så lokalt max.

f) $f(x,y,z) = (1+x^2)e^{-y^2-z^2}$
 $f(0,0,0) = 1$, $f(x,0,0) = 1+x^2 > 1$, $x \neq 0$
 $f(0,y,0) = e^{-y^2} < 1$ om $y \neq 0$
så ej extrempunkt

g) $f(x,y) = (x+y)^2 + xy^3$, $f(0,0) = 0$.
 $f(x,0) = x^2 > 0$ om $x \neq 0$, $f(x,-x) = -x^4 < 0$, $x \neq 0$
så ej extrempunkt

h) $f(x,y) = \begin{cases} y^2 \arctan \frac{x}{y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$

$f(-t,t) = t^2 \arctan(-1/t) < 0$ om $t > 0$

$f(t,t) = t^2 \arctan(1/t) > 0$ om $t > 0$

så ej lok. extrempunkt

i) $f(x,y) = (x-y)^2 + xy^3 = x^2 + y^2 - 2xy + xy^3$
 $= (x^2 + y^2 + xy(y^2-2)) = \left(x + \frac{y(y^2-2)}{2}\right)^2 - \frac{y^2(y^2-2)^2}{4} + y^3$
 $= \left(x + \frac{y(y^2-2)}{2}\right)^2 + y^2 \left(\frac{4 - y^4 + 4y^2 - 4}{4}\right) \geq 0$
 > 0 om y nära 0, lokalt min.

2.64)

$$a) (x^2 + y^2 - 1) e^y = (x^2 + y^2 - 1) \left(1 + y + \frac{y^2}{2} + O(y^3)\right) =$$

$$= -1 - y + x^2 + y^2 - \frac{y^2}{2} + O(y^3) = -1 - y + x^2 + \frac{y^2}{2} + O(|(x,y)|^3)$$

$O(|(x,y)|^3)$

$$b) 2\sqrt{1+x^2+y^2} - \cos(x-2) - y =$$

$$= \sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + O(t^3)$$

$$\cos s = 1 - \frac{s^2}{2} + O(s^4)$$

$$= 2\left(1 + \frac{(x^2+y^2)}{2} - \frac{(x^2+y^2)^2}{8} + O((x^2+y^2)^3)\right) -$$

$$- \left(1 - \frac{(x-2)^2}{2} + O((x-2)^4)\right) - y =$$

$$= \dots = 1 + \frac{3x^2}{2} - x^2 - \frac{y^2}{4} + \frac{y^2}{2} + O(|(x,y,2)|^3)$$

2.65) $f(x,y) = \ln(2x^2 + xy + y^2)$, $f(2,-1) = \ln 7$

$$f'_x = \frac{4x+y}{2x^2+xy+y^2}, \quad f'_x(2,-1) = \frac{8-1}{8+2+1} = 1$$

$$f'_y = \frac{x+2y}{2x^2+xy+y^2}, \quad f'_y(2,-1) = \frac{2-2}{8+2+1} = 0$$

$$f''_{xx} = \frac{4(2x^2+xy+y^2) - (4x+y)(4x+y)}{(2x^2+xy+y^2)^2}, \quad f''_{xx}(2,-1) = \dots = -\frac{3}{7}$$

$$f''_{xy} = \frac{(2x^2+xy+y^2) - (4x+y)(x+2y)}{(2x^2+xy+y^2)^2}, \quad f''_{xy}(2,-1) = \dots = \frac{1}{7}$$

$$f''_{yy} = \frac{2(2x^2+xy+y^2) - (x+2y)(x+2y)}{(2x^2+xy+y^2)^2}, \quad f''_{yy}(2,-1) = \dots = \frac{2}{7}$$

$$f(2+h, -1+k) = f(2,-1) + f'_x(2,-1)h + f'_y(2,-1)k + \frac{f''_{xx}(2,-1)}{2}h^2 +$$

$$+ f''_{xy}(2,-1)hk + \frac{f''_{yy}(2,-1)}{2}k^2 + O(|(h,k)|^3)$$

$$= \ln 7 + h - \frac{3h^2}{14} + \frac{hk}{7} + \frac{k^2}{7} + O(|(h,k)|^3)$$

2.66)

$$a) h^2 - hk + k^2 = \left(h - \frac{k}{2}\right)^2 - \frac{k^2}{4} + k^2 = \left(h - \frac{k}{2}\right)^2 + \frac{3k^2}{4}$$

pos. def.

b)

$$h^2 + hk - k^2 = (h \ k) \begin{pmatrix} 1 & 1/2 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1/2 \\ 1/2 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - \frac{1}{4} = \lambda^2 - \frac{5}{4} = 0 \quad \lambda = \pm \frac{\sqrt{5}}{2}$$

indefinit

c)

$$hk = Q(h, k) \quad Q(h, h) = h^2 > 0, \quad h \neq 0, \quad Q(h, -h) = -h^2 < 0, \quad h \neq 0$$

indefinit

d)

$$6k^2 + l^2 - 6hk + 2hl + 4kl = (l - h + 2k)^2 - h^2 + 4hk - 4k^2 + 6k^2 - 6hk - 2l^2$$

$$= (l - h + 2k)^2 - (h + k)^2 + k^2 + 2k^2 =$$

$$= (l - h + 2k)^2 - (h + k)^2 + 3k^2 \quad \text{indefinit}$$

e)

$$4hk + 4kl - 2h^2 - 3k^2 - 4l^2 = (h \ k \ l) \begin{pmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{pmatrix} \begin{pmatrix} h \\ k \\ l \end{pmatrix}$$

$$\begin{vmatrix} -2-\lambda & 2 & 0 \\ 2 & -3-\lambda & 2 \\ 0 & 2 & -4-\lambda \end{vmatrix} = (-2-\lambda)((-3-\lambda)(-4-\lambda) - 4) - 2(2(-4-\lambda))$$

$$= (-2-\lambda)(\lambda^2 + 7\lambda + 8) + 4\lambda + 16 =$$

$$= -\lambda^3 - 9\lambda^2 - 28\lambda = 0 \quad (\Leftrightarrow) \quad \lambda = 0 \quad \text{oder} \quad \lambda^2 + 9\lambda + 28 = 0$$

$$(\Leftrightarrow) \quad \lambda = 0 \quad \text{oder} \quad \lambda = -\frac{9}{2} \pm \sqrt{\frac{81}{4} - 18} = -\frac{9}{2} \pm \sqrt{\frac{9}{4}} = -\frac{9}{2} \pm \frac{3}{2}$$

$\lambda = 0, -3, -6$ Negativ semidefinit

$$-14 - 43 = -57 \quad \lambda = -7 = 0$$

$$-14 - 43 = -57$$

2.66)

f) ~~$h^2 + 2hk + k^2 = (h+k)^2$~~

pos. semidefinit.

$$Q(h, k, l) = h^2 + 2hk + 2k^2 = (h+k)^2 + k^2 \geq 0$$

 $Q(0, 0, l) = 0$ för alla l

g) $(h-k)^2 + (k-l)^2 + (l-h)^2 \geq 0$

$$Q(h, h, h) = 0^2 + 0^2 + 0^2 = 0$$

pos. semidefinit.

h)

$$h^2 + 2hk + 3k^2 - 4kl + 6l^2 =$$

$$= (h+k)^2 + 2k^2 - 4kl + 6l^2 = (h+k)^2 + 2(k-l)^2 + 4l^2$$

pos. definit.

i)

$$h^2 + 2hk + k^2 - 12kl = (h+k)^2 - 12kl$$

$$Q(h, 0, 0) = h^2 > 0 \text{ om } h \neq 0$$

$$Q(h, -h, -h) = (h-h)^2 - 12h^2 = -12h^2 < 0 \text{ om } h \neq 0 \text{ indefinit}$$

2.68

a)

$$f(x, y) = -1 - y + \underbrace{x^2 - \frac{y^2}{2}}_{\text{indefinit}} + O(\|(x, y)\|^3)$$

~~indefinit~~ \Rightarrow sadelpunkt $f'_y(0, 0) = -1 \neq 0$, så
ej stationär

b) $f(x, y, z) = 1 + \frac{3x^2}{2} - xz - \frac{y^2}{4} + \frac{z^2}{2} + O(\|(x, y, z)\|^3)$

Indefinit \Rightarrow sadelpunkt

2.69)

a)

$$f(x,y) = -1 - \underbrace{x^2 + 2xy - 2y^2}_{\text{andragradsform}} + x^3y - x^4 = -1 - x^2 + 2xy - 2y^2 + O(\|(x,y)\|^3)$$

$$-x^2 + 2xy - 2y^2 = -(x-y)^2 - y^2 \quad \text{neg. definit}$$

\Rightarrow lokalt max.

b)

$$f(x,y) = x^2 + x^3 - 2xy + y^2 = (x-y)^2 + x^3$$

$f(x,x) = x^3$ byter tecken i 0, så varken eller.

c)

$$f(x,y) = x^2 + x^4 - 2xy + y^2 = (x-y)^2 + x^4 \geq 0$$

$$f(0,0) = 0 \quad \Rightarrow \text{min.}$$

d)

$$f(x,y,z) = x + y^2 - y^3 + z^2$$

$$\nabla f(0,0,0) = (1, 0, 0) \neq (0, 0, 0) \quad \text{ej extrempunkt}$$

e)

$$f(x,y,z) = 2 \cos(x+y+z) + e^{xy} + e^{yz} + e^{xz}$$

$$= 2 \left(1 - \frac{(x+y+z)^2}{2} + O(\|(x,y,z)\|^4) \right) +$$

$$+ \left(1 + xy + O(x^2y^2) \right) + \left(1 + yz + O(y^2z^2) \right) + \left(1 + xz + O(x^2z^2) \right)$$

$$= 5 - (x+y+z)^2 + xy + xz + yz + O(\|(x,y,z)\|^4)$$

$$= \dots = 5 - \underbrace{\left(x + \frac{y}{2} + \frac{z}{2} \right)^2}_{\text{Neg. definit}} - \frac{3}{4} \left(y + \frac{z}{3} \right)^2 - \frac{2z^2}{3} + O(\|(x,y,z)\|^4)$$

Neg. definit

\Rightarrow lokalt max.

f)

$$f(x,y,z) = \cos(x+y+z) + \cos x \leq 2 \quad f(0,0,0) = 2$$

$$= \cos(x+y+z)$$

Max.

$$= 2 - x^2$$

$$= 2 - x^2 - \frac{1}{2}(y+z)^2 - \frac{1}{4}(y-z)^2$$

$$= 2 - x^2 - \frac{1}{2}(y+z)^2 - \frac{1}{4}(y-z)^2$$

2.70)

a)

$$f(x, y) = 3 + 4x - 4y - x^2 - 2y^2$$

$$\nabla f = (4 - 2x, -4 - 4y) = (0, 0) \Leftrightarrow x = 2, y = -1$$

$$f''_{xx} = -2, \quad f''_{xy} = 0, \quad f''_{yy} = -4.$$

$$f(2+h, -1+k) = 9 - \underbrace{h^2 - 2k^2}_{\text{neg. definit}}$$

lokal max i (2, -1).

$$b) f(x, y, z) = x^2 + y^2 + z^2 - xy + 2z + x$$

$$\nabla f = (2x - y + 1, 2y - x, 2z + 2) = (0, 0, 0)$$

$$\Leftrightarrow \begin{cases} 2x - y = -1 \\ 2y - x = 0 \\ z = -1 \end{cases} \Leftrightarrow \begin{cases} 3y = -1 \Rightarrow y = -1/3 \\ x = 2y = -2/3 \\ z = -1 \end{cases}$$

$$f''_{xx} = 2, \quad f''_{xy} = -1, \quad f''_{yy} = 2, \quad f''_{xz} = 0, \quad f''_{yz} = 0, \quad f''_{zz} = 2$$

$$h^2 - hk + k^2 + l^2 = \left(h - \frac{k}{2}\right)^2 + \frac{3k^2}{4} + l^2 \quad \text{pos. definit}$$

lokal min i $(-2/3, -1/3, -1)$.

c)

$$f(x, y) = x e^{-2x^2 - y^2}$$

$$\nabla f = (e^{-2x^2 - y^2} - 4x^2 e^{-2x^2 - y^2}, -2xy e^{-2x^2 - y^2}) = (0, 0)$$

$$\Leftrightarrow \begin{cases} 1 - 4x^2 = 0 \\ -2xy = 0 \end{cases} \Leftrightarrow \begin{cases} x = \pm 1/2 \\ y = 0 \end{cases}$$

$$f''_{xx} = -4x \cdot e^{-2x^2 - y^2} - 8x^2 e^{-2x^2 - y^2} + 16x^3 e^{-2x^2 - y^2}$$

$$f''_{xy} = -2y e^{-2x^2 - y^2} + 8xy e^{-2x^2 - y^2}$$

$$f''_{yy} = -2x e^{-2x^2 - y^2} + 4xy^2 e^{-2x^2 - y^2}$$

$$(1/2, 0) \quad f''_{xx}(1/2, 0)h^2 + 2f''_{xy}(1/2, 0)hk + f''_{yy}(1/2, 0)k^2 =$$

$$= e^{-1/2} \left(-\frac{4}{8}h^2 - k^2 \right) \quad \text{neg. def} \Rightarrow \text{lokal max}$$

$$(-1/2, 0) \quad f''_{xx}(-1/2, 0)h^2 + 2f''_{xy}(-1/2, 0)hk + f''_{yy}(-1/2, 0)k^2 =$$

$$= e^{-1/2} \left(\frac{4}{8}h^2 + k^2 \right) \quad \text{pos. def} \Rightarrow \text{lokal min.}$$

2.70)

d)

$$f(x,y) = x+y - 3 \ln(2+xy) \quad x > 0, y > 0$$

$$\nabla f = \left(1 - \frac{3y}{2+xy}, 1 - \frac{3x}{2+xy} \right) = (0,0)$$

$$\Leftrightarrow \begin{cases} 3y = 2+xy \\ 3x = 2+xy \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \\ 3x = 2+x^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2} \\ y = x \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y = 2 \\ x = y = 1 \end{cases}$$

$$f''_{xx} = \frac{3y^2}{(2+xy)^2}, \quad f''_{xy} = \frac{-3(2+xy) - 3y \cdot x}{(2+xy)^2}, \quad f''_{yy} = \frac{3x^2}{(2+xy)^2}$$

$$\begin{aligned} (1,1): \quad & f''_{xx}(1,1)h^2 + 2f''_{xy}(1,1)hk + f''_{yy}(1,1)k^2 = \\ & = \frac{1}{3}h^2 + \frac{4}{3}hk + \frac{1}{3}k^2 = \frac{1}{3}(h^2 + 4hk + k^2) = \\ & = \frac{1}{3}((h+2k)^2 - 3k^2) \quad \text{indefinit} \Rightarrow \text{sattelpunkt.} \end{aligned}$$

$$\begin{aligned} (2,2): \quad & f''_{xx}(2,2)h^2 + 2f''_{xy}(2,2)hk + f''_{yy}(2,2)k^2 = \\ & = \frac{1}{3}h^2 - \frac{1}{3}hk + \frac{1}{3}k^2 = \frac{1}{3}(h^2 + 4k + k^2) = \\ & = \frac{1}{3}\left(\left(h - \frac{k}{2}\right)^2 + \frac{3k^2}{4}\right) \quad \text{pos. definit} \\ & \quad \text{lokal min.} \end{aligned}$$

e)

$$f(x,y) = \ln(x^2+y^2) - x - 2y$$

$$\nabla f = \left(\frac{2x}{x^2+y^2} - 1, \frac{2y}{x^2+y^2} - 2 \right) = (0,0)$$

$$\Leftrightarrow \begin{cases} x^2+y^2 = 2x \\ x^2+y^2 = 4y \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x = y \\ x^2 + 4x^2 = 2x \end{cases}$$

$$\Leftrightarrow \begin{cases} (x=y=0) \text{ ej i } D_f \\ x = 2/5, y = 4/5 \end{cases}$$

$$f''_{xx} = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2}, \quad f''_{xy} = \frac{-2x \cdot 2y}{(x^2+y^2)^2}, \quad f''_{yy} = \frac{2(x^2+y^2) - 2y \cdot 2y}{(x^2+y^2)^2}$$

$$\begin{aligned} & f''_{xx}(2/5, 4/5)h^2 + 2f''_{xy}(2/5, 4/5)hk + f''_{yy}(2/5, 4/5)k^2 = \\ & = \frac{24}{20^2}h^2 - \frac{64hk}{20^2} - \frac{24}{20^2}k^2 = \frac{8}{20^2}(3h^2 - 8hk - 3k^2) \quad \text{indef.} \\ & \quad \text{sattelpunkt} \end{aligned}$$

2.70) f)

$$f(x, y, z) = x^4 + y^4 + z^4 - 4xyz$$

$$\nabla f = (4x^3 - 4yz, 4y^3 - 4xz, 4z^3 - 4xy) = (0, 0, 0)$$

$$\Leftrightarrow \begin{cases} x^3 = yz \\ y^3 = xz \\ z^3 = xy \end{cases}$$

$$x=0 \Rightarrow y=z=0 \quad \text{d.s.m. s\u00e4 } (0,0,0) \text{ \u00e4r punkt}$$

$$x, y, z \neq 0 \quad \text{g\u00e4r}$$

$$\begin{cases} \frac{x^3}{y^3} = \frac{y}{x} \Rightarrow x^4 = y^4 \\ \frac{y^3}{z^3} = \frac{z}{y} \Rightarrow y^4 = z^4 \\ z^3 = xy \end{cases}$$

$$x = \pm y, y = \pm z \quad \text{samt } z^3 = xy \quad \text{g\u00e4r nu}$$

m\u00f6jligheter \u00e4r $(1,1,1), (1,-1,-1), (-1,-1,1), (-1,1,-1), (0,0,0)$

$$f''_{xx} = 12x^2, \quad f''_{yy} = 12y^2, \quad f''_{zz} = 12z^2, \quad f''_{xy} = -4z, \quad f''_{xz} = -4y, \quad f''_{yz} = -4x$$

$(0,0,0)$: Alla andra derivator n\u00e4r, men

$$f(x, x, x) = 3x^4 - 4x^3 < 0 \quad \text{n\u00e4r } 0, \quad x \neq 0$$

$$f(x, 0, 0) = x^4 > 0 \quad \text{n\u00e4r } 0, \quad x \neq 0 \quad \text{s\u00e4 s\u00e4delpunkt}$$

$$Q(h, k, l) = f''_{xx} h^2 + 2f''_{xy} hk + f''_{xz} hl + f''_{yy} k^2 + 2f''_{yz} kl + f''_{zz} l^2$$

$$(1,1,1) \quad Q(h, k, l) = 12h^2 - 8hk - 8hl + 12k^2 - 8kl + 12l^2$$

$$= 4(3h^2 - 2hk - 2hl + 3k^2 - 2kl + 3l^2) =$$

$$= 4 \left(3 \left(h - \frac{k}{3} - \frac{l}{3} \right)^2 + \frac{8k^2}{3} - 4kl + \frac{8l^2}{3} \right)$$

$$= 4 \left(3 \left(h - \frac{k}{3} - \frac{l}{3} \right)^2 + \frac{8}{3} \left(k - \frac{2l}{6} \right)^2 + \frac{7l^2}{6} \right)$$

Pos. definit. Lokalt min i $(1,1,1)$.

\u00d6vriga punkter behandlas analogt, och alla

dess\u00e4r \u00e4r lokala min.

2.70)

9) $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

$\nabla f = (3x^2 + 3y^2 - 15, 6xy - 12) = (0,0)$

$\Leftrightarrow \begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases} \Leftrightarrow \begin{cases} x^2 + 4/x^2 = 5 \\ y = 2/x \end{cases} \Leftrightarrow \begin{cases} x^4 - 5x^2 + 4 = 0 \\ y = 2/x \end{cases}$

$\Leftrightarrow \begin{cases} x^2 = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} = \frac{5}{2} \pm \frac{3}{2} \\ y = 2/x \end{cases} \Leftrightarrow \begin{cases} X=2, y=1, X=-2, y=-1 \\ X=1, y=2, X=-1, y=-2 \end{cases}$

$f''_{xx} = 6x, f''_{yy} = 6y, f''_{xy} = 6x$

$Q(h,k) = f''_{xx}h^2 + 2f''_{xy}hk + f''_{yy}k^2$

i) (2,1) $Q(h,k) = 12h^2 + 12hk + 12k^2 = 12(h^2 + hk + k^2) = 12((h - \frac{k}{2})^2 + \frac{3k^2}{4})$

pos. det. lok. min.

i) (-2,-1) $Q(h,k) = -12h^2 - 12hk - 12k^2 \Rightarrow$ neg. det. lok. max.

i) (1,2) : $Q(h,k) = 6h^2 + 24hk + 6k^2 = 6(h^2 + 4hk + k^2) = 6((h+2k)^2 - 3k^2)$
indef. sattel.

i) (-1,-2) : $Q(h,k) = -6h^2 - 24hk - 6k^2 = -6(h^2 + 4hk + k^2)$
indefinit, sattel.

h) $f(x,y) = (x^2 + y^2 - 4)e^{-x-y}$

$\nabla f = (2xe^{-x-y} - (x^2 + y^2 - 4)e^{-x-y}, 2ye^{-x-y} - (x^2 + y^2 - 4)e^{-x-y}) = (0,0)$

$\Leftrightarrow \begin{cases} x^2 + y^2 - 2x = 4 \\ x^2 + y^2 - 2y = 4 \end{cases} \Leftrightarrow \begin{cases} x = y \\ 2x^2 - 2x = 4 \end{cases} \Leftrightarrow \begin{cases} x = y \\ x^2 - x - 2 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = y \\ x = \frac{1}{2} \pm \frac{3}{2} \end{cases} \quad (-1,-1) \text{ oder } (2,2) \text{ start-punkten}$

$f''_{xx} = (2 - 2x - 2x + (x^2 + y^2 - 4))e^{-x-y}, f''_{yy} = (-2x - 2y + (x^2 + y^2 - 4))e^{-x-y}$

$f''_{yy} = (2 - 2y - 2y + (x^2 + y^2 - 4))e^{-x-y}$

i) (-1,-1): $e^2(4h^2 + 4hk + 4k^2) = 4e^2(h^2 + hk + k^2) = 4e^2((h + \frac{k}{2})^2 + \frac{3k^2}{4})$
pos. det. \Rightarrow lok. min.

i) (2,2) $e^4(-2h^2 - 8hk - 2k^2) = -2e^4((h+2k)^2 - 3k^2)$ indef. sattel.

2.70 i)

$$f(x, y) = 4x^2 + 4xy^2 + y^4 + y^6$$

$$\nabla f = (8x + 4y^2, 8xy + 4y^3 + 6y^5) = (0, 0) \Leftrightarrow$$

$$\begin{cases} 2x + y^2 = 0 \end{cases}$$

$$\begin{cases} 4xy + 2y^3 + 3y^5 = 0 \Leftrightarrow y = 0 \text{ eller } 4x + 2y^2 + 3y^4 = 0 \end{cases}$$

$$(x, y) = (0, 0) \text{ eller } \begin{cases} y^2 = -2x \\ 4x + 2(-2x) + 3(-2x)^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y^2 = -2x \\ 12x^2 = 0 \end{cases}$$

(0,0) enda stationära punkter.

Andaderivatstest, ger inset då kv. formen

blir positivt semidefinit.

$$4x^2 + 4xy^2 + y^4 + y^6 = (2x + y^2)^2 + y^6 \geq 0 = f(0, 0)$$

Så minimum i (0,0).

j)

$$f(x, y, z) = 4x + y^2/x + 4z^2/y + 8/z \quad (x, y, z \neq 0)$$

$$\nabla f = (4 - y^2/x^2, 2y/x - 4z^2/y^2, 8z/y - 8/z^2) = (0, 0, 0)$$

$$\Leftrightarrow \begin{cases} 4x^2 = y^2 \\ 2y^2 = 4xz^2 \\ 8z^3 = 8y \end{cases} \quad \begin{array}{l} \text{Andra ekv. ger att om} \\ x \text{ är negativt måste } y \text{ vara negativt.} \end{array}$$

$$\text{Så } \begin{cases} y = 2x \\ 8x^3 = 2xz^2 \\ 8z^3 = 2x \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ z^2 = 4x^2 \\ z^3 = 2x \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ z = \pm 2x \\ \pm 8x^3 = 2x \end{cases}$$

$$\begin{cases} x = 1/2 \\ y = 1 \\ z = 1 \end{cases} \text{ eller } \begin{cases} x = -1/2 \\ y = -1 \\ z = -1 \end{cases}$$

$$f''_{xx} = \frac{2y^2}{x^3}, \quad f''_{xy} = \frac{-2y}{x^2}, \quad f''_{xz} = 0, \quad f''_{yy} = \frac{2}{x} + \frac{8z^2}{y^3}, \quad f''_{yz} = \frac{-8z}{y^2}, \quad f''_{zz} = \frac{8}{z^3} + \frac{8}{z^3}$$

$$(1/2, 1, 1): 16h^2 - 6hk + 9k^2 - 16kl + 24l^2 = 16(h - \frac{15}{2})^2 + 5(k - \frac{81}{5})^2 + \frac{56l^2}{5}$$

pos def. \Rightarrow lokalt min

$$(-1/2, -1, -1): -16h^2 + 16hk - 9k^2 + 16kl - 24l^2 \Rightarrow \text{neg. definit. lok. max.}$$

2.63)

$$f(x,y) = (y-x^2)(y-3x^2)$$

$$a) f(x, kx) = (kx-x^2)(kx-3x^2) =$$

$$= k^2 x^2 + O(x^3) \quad \text{für } k \neq 0 \text{ lok. min. i. } x=0$$

$$f(x,0) = 3x^4 \text{ lok. min. i. } y=0$$

$$f(0,y) = y^2 \text{ lok. min. i. } y=0$$

$$b) f(x, 2x^2) = (2x^2-x^2)(2x^2-3x^2) = -x^4 \text{ lok. max. i. } x=0$$

2.67)

$$h^2 + 2ahk + k^2 = (h+ak)^2 + (1-a^2)k^2$$

$$1-a^2 > 0 \Rightarrow \text{pos. definit}$$

$$1-a^2 = 0 \Rightarrow \text{pos. semidefinit}$$

$$1-a^2 < 0 \Rightarrow \text{indefinit}$$

2.71) $f(x,y) = 4x^2 e^y - 2x^4 - e^{4y}$

$$a) \nabla f = (8x e^y - 8x^3, 4x^2 e^y - 4e^{4y}) = (0,0)$$

$$\Leftrightarrow \begin{cases} 8x e^y = 8x^3 \\ 4x^2 e^y = 4e^{4y} \end{cases} \Leftrightarrow \begin{cases} e^y = x^2 \\ e^{2y} = e^{4y} \end{cases} \Leftrightarrow \begin{cases} x = \pm 1 \\ y = 0 \end{cases}$$

b)

$$f''_{xx} = 8e^y - 24x^2, \quad f''_{xy} = 8x e^y, \quad f''_{yy} = 4x^2 e^y - 16e^{4y}$$

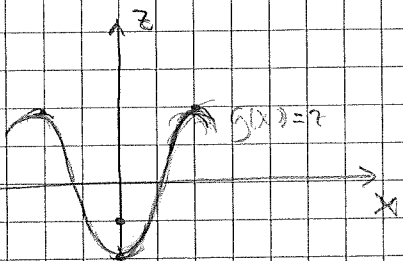
$$\text{i. } (1,0): -16h^2 + 16hk - 12k^2 = -4(4h^2 - 4hk + 3k^2) =$$

$$= -4\left(4\left(h - \frac{k}{2}\right)^2 + 2k^2\right) \quad \text{neg. definit} \Rightarrow \text{lok. max}$$

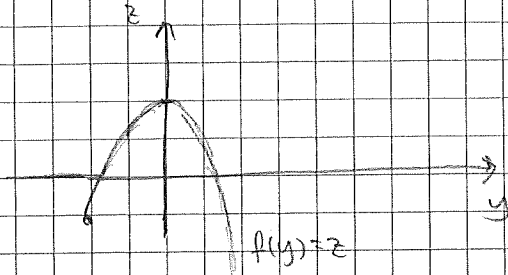
$$\text{i. } (-1,0): -16h^2 - 16hk - 12k^2 = -4(4h^2 + 4hk + 3k^2) =$$

$$= -4\left(4\left(h + \frac{k}{2}\right)^2 + 2k^2\right) \quad \text{neg. definit} \Rightarrow \text{lok. max}$$

c) $g(x) = f(x,0) = 4x^2 - 2x^4 - 1$



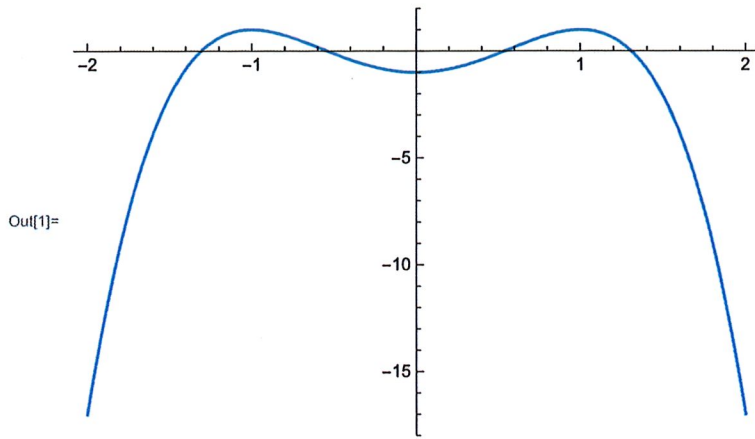
d) $h(y) = f(\pm 1, y) = 4e^y - 2e^{4y}$



2.71

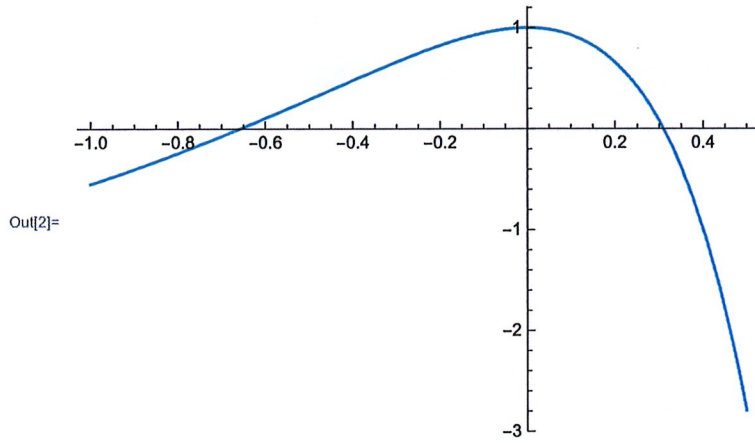
In[1]:= `Plot[4 * x^2 - 2 * x^4 - 1, {x, -2, 2}]`

c)



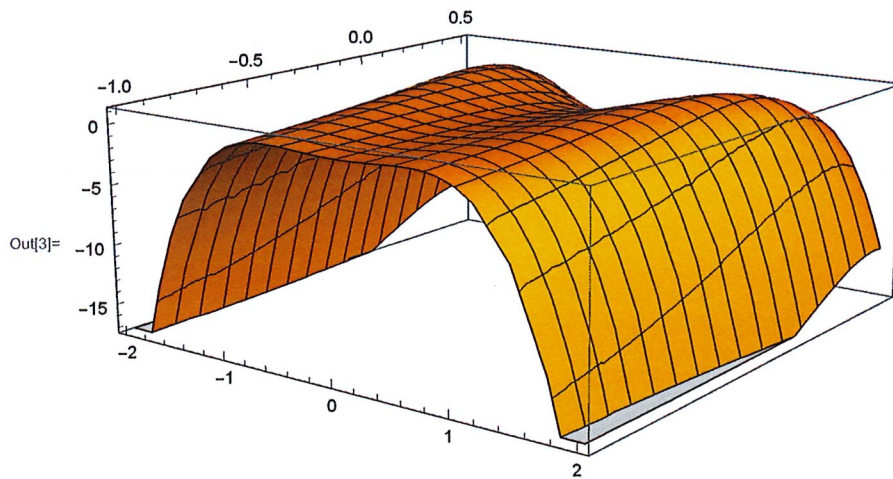
In[2]:= `Plot[4 * Exp[y] - 2 - Exp[4 * y], {y, -1, 0.5}]`

d)



In[3]:= `Plot3D[4 * x^2 * Exp[y] - 2 * x^4 - Exp[4 * y], {x, -2, 2}, {y, -1, 0.5}]`

e)



$$2.73) \quad f(x,y) = \begin{cases} \frac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Eftersom $f(x,y) \geq 0$ med likhet om och endast om $x=y=0$ har funktionen ett (globalt) strängt min i $(x,y) = (0,0)$.

Om $(x,y) \neq (0,0)$ kan vi skriva

$$f(\rho, \theta) = \frac{\rho^4(\cos^4\theta + \sin^4\theta)}{\rho^2} = \rho^2(\cos^4\theta + \sin^4\theta)$$

(Om $f(x,y)$ har ett lokalt max/min i punkten (a,b) har $f(\rho, \theta)$ det motsvarande punktet (ρ, θ) i ρ -planet).

Eftersom $f'_\rho = 2\rho(\cos^4\theta + \sin^4\theta) > 0$ om $\rho > 0$

har funktionen inga stationära punkter i $(x,y) \neq (0,0)$.

2.74) Med $(a,b) = (0,0)$

$$f(x,y) = x^2 + 100xy + y^2$$

$$\text{gäller } f''_{xx} = 2, \quad f''_{xy} = 100, \quad f''_{yy} = 2$$

$$f(x,x) = 102x^2 > 0 \quad x \neq 0$$

$$f(x,-x) = -98x^2 < 0, \quad x \neq 0 \quad \text{s\u00e4 saddlepunkt.}$$