Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2020-10-30 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad .$$

Determine the general solution to this system and draw a picture of the entire phase plane (including negative x and y). Is the steady state (= equilibrium point) $(0, 0)$ stable?
What is the solution if the initial values are $x(0) = 3$ and $y(0) = 2$?
Show that the level of $y(t)$ is approximately twice the level of $x(t)$ for large t (for any initial levels $x(0) > 0$, $y(0) > 0$).

2. The population $N_n \geq 0$ in a time discrete population model satisfies

$$N_{n+1} = 2N_n e^{-N_n} \quad , \; n = 0, 1, 2, \ldots$$

Find the steady states (= equilibrium points) of the model and determine their stability.
What happens to the population as $n \to \infty$ if $N_0 > 0$?
Sketch a cobweb diagram for some N_0. (For this you need to sketch the graph of $f(x) = 2xe^{-x}$ for $x \geq 0$; useful is to know $f(0)$, $\lim_{x\to\infty} f(x)$, and where $f(x)$ has its maximum)

3. A model for a phytoplankton-herbivore system is given by the equations

$$\begin{cases} \dfrac{dx}{dt} = x + x^2 - xy \\ \dfrac{dy}{dt} = 3xy - 2y^2 \end{cases}$$

Here $x(t)$ is phytoplankton density and $y(t)$ herbivore density.
Find all steady states and determine their stability. Draw, for $x \geq 0$ and $y \geq 0$, a phase plane picture (with nullclines and directions of the vector field).
After a period in which both densities have decreased, which of the two densities begins to increase first?

PLEASE TURN

4. Let S_n, I_n and R_n be the number of susceptibles, infective, and removed, respectively, at time $n \geq 0$ in a time discrete SIRS epidemic model. The total population N is constant, $N = 290$, and with $R_n = N - S_n - I_n$ it is sufficient to study a two-dimensional system for S_n and I_n. With $0 < \alpha < 1$ constant and certain values of the other parameters, the system is

$$\begin{cases} S_{n+1} = S_n - \dfrac{S_n I_n}{300} + \frac{5}{6}(290 - S_n - I_n) \\ I_{n+1} = I_n - \alpha I_n + \dfrac{S_n I_n}{300} \end{cases}$$

For what value of α is there a steady state (= equilibrium point) $(\bar{S}, \bar{I})$ with the number of susceptibles $\bar{S} = 200$? What is the number of infective $\bar{I}$ at this steady state? Show that the steady state is stable.

5. For $0 < x < \pi$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} u_t = u_{xx} + u \\ u(t, 0) = 0 \\ u(t, \pi) = 0 \\ u(0, x) = x \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

What is the limit of $u(t, x)$ as $t \to \infty$?

6. A model with spatial diffusion for two populations $N(t, x)$ and $P(t, x)$ that compete for the same resources is given by

$$\begin{cases} N_t = N(6 - 3N - P) + D_1 N_{xx} \\ P_t = P(4 - P - N) + D_2 P_{xx} \end{cases}$$

Find the four spatially uniform steady states, and show that only one of them is stable in absence of diffusion ($D_1 = D_2 = 0$).
Show that diffusive instability is impossible in this model, so perturbations cannot give rise to spatial patterns in the populations.

TATM38, 30/10 2020, solution sketches

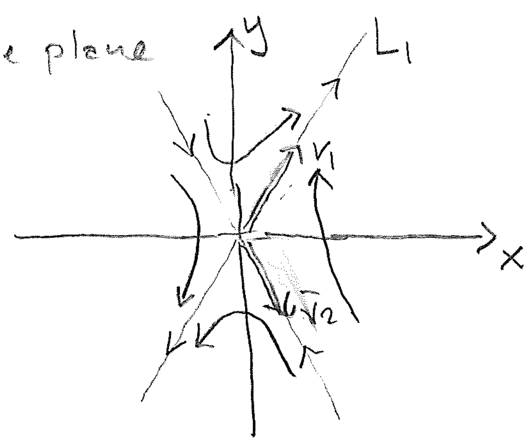
① $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ Eigenvalues $\begin{vmatrix} -1-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 4 = 0 \Rightarrow$

$\left.\begin{matrix} \lambda_1 = 1 > 0 \\ \lambda_2 = -3 < 0 \end{matrix}\right\} \Rightarrow$ saddle point (unstable) at $(0,0)$

Eigenvectors $\lambda_1 = 1 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = -3 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow$ the general solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow c_1 = 2, c_2 = 1 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2e^t + e^{-3t} \\ 4e^t - 2e^{-3t} \end{pmatrix}$

Phase plane



Any solution curve with $x(0)>0, y(0)>0$ will approach the line L_1 ($y=2x$) upwards $\Rightarrow y(t) \approx 2x(t)$ for large t

$\left(\text{or } e^{-3t} \to 0 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow y \approx 2x\right)$

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② $N_{n+1} = f(N_n) = 2N_n e^{-N_n}$

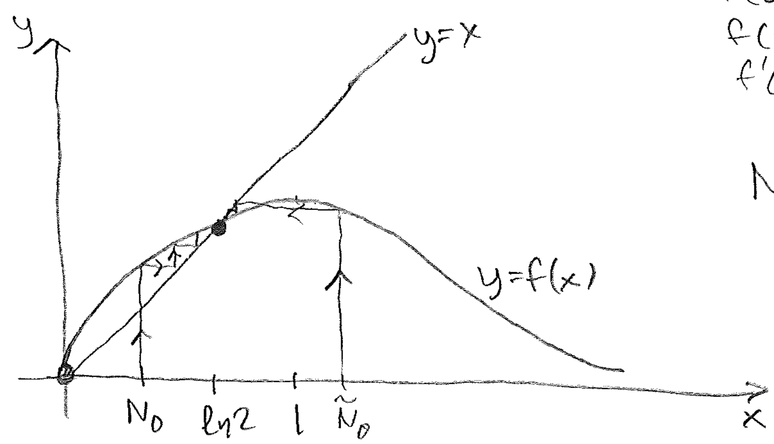
Steady states $\bar{N} = f(\bar{N}) \Rightarrow \bar{N} = 2\bar{N}e^{-\bar{N}} \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = \ln 2$

$f'(N) = 2(1-N)e^{-N} \Rightarrow$

$|f'(0)| = 2 > 1 \Rightarrow \bar{N}_1 = 0$ unstable

$|f'(\ln 2)| = |2(1-\ln 2)\cdot \frac{1}{2}| = 1 - \ln 2 < 1 \Rightarrow \bar{N}_2 = \ln 2$ stable

Cobweb



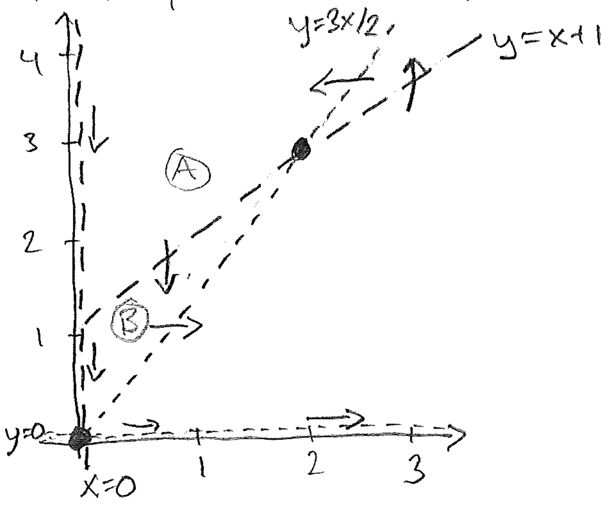
$f'(x) = 0$ at $x = 1$ (max point)
$f(0) = 0$
$f(x) \to 0, x \to \infty$
$f'(0) = 2$

$N_n \to \bar{N}_2 = \ln 2, n \to \infty$

if $N_0 > 0$

③ $\begin{cases} x' = x(1+x-y) \\ y' = y(3x-2y) \end{cases}$ x-nullclines: $x=0$ and $y=x+1$, 2 lines
y-nullclines: $y=0$ and $y=\frac{3x}{2}$, 2 lines

Phase plane (for $x\geq 0, y\geq 0$)



2 steady states (intersections of x-nullclines and y-nullclines with $x\geq 0, y \geq 0$):

$(\bar{x}_1, \bar{y}_1) = (0,0)$

$(\bar{x}_2, \bar{y}_2) = (2,3)$

$$J(x,y) = \begin{pmatrix} 1+2x-y & -x \\ 3y & 3x-4y \end{pmatrix} \Rightarrow$$

$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with $\lambda_1 = 1 > 0, \lambda_2 = 0 \Rightarrow (0,0)$ unstable

$J(2,3) = \begin{pmatrix} 2 & -2 \\ 9 & -6 \end{pmatrix} = J \Rightarrow \begin{cases} Tr\, J = -4 < 0 \\ \det J = 6 > 0 \end{cases} \Rightarrow (2,3)$ stable [or $\lambda_{1,2} = -2 \pm i\sqrt{2}$ $\Rightarrow Re\, \lambda_{1,2} < 0 \Rightarrow$ stable]

Both densities decrease in region Ⓐ in the phase plane, from Ⓐ one moves to Ⓑ, where y is still decreasing but x is increasing $\Rightarrow$ x increases first

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④ $\begin{cases} S_{n+1} = S_n - \frac{1}{300} S_n I_n + \frac{5}{6}(290 - S_n - I_n) \\ I_{n+1} = I_n - \alpha I_n + \frac{1}{300} S_n I_n \end{cases}$ Steady states $(\bar{S}, \bar{I})$ satisfy $S_{n+1} = S_n = \bar{S}$, $I_{n+1} = I_n = \bar{I}$

$\Rightarrow \begin{cases} 0 = -\frac{1}{300}\bar{S}\bar{I} + \frac{5}{6}(290 - \bar{S} - \bar{I}) \\ 0 = -\alpha \bar{I} + \frac{1}{300}\bar{S}\bar{I} \end{cases}$ If $\bar{S} = 200$: $\begin{cases} -\frac{2\bar{I}}{3} + \frac{5}{6}(90 - \bar{I}) = 0 & (1) \\ (-\alpha + \frac{2}{3})\bar{I} = 0 & (2) \end{cases}$

$(1) \Rightarrow \bar{I} = 50 \underset{(2)}{\Rightarrow} \alpha = \frac{2}{3}$

$$J(S,I) = \begin{pmatrix} 1 - \frac{I}{300} - \frac{5}{6} & -\frac{S}{300} - \frac{5}{6} \\ \frac{I}{300} & 1 - \alpha + \frac{S}{300} \end{pmatrix} \underset{\alpha = \frac{2}{3}}{\Longrightarrow} J(200,50) = \begin{pmatrix} 0 & -\frac{3}{2} \\ \frac{1}{6} & 1 \end{pmatrix} = J$$

Eigenvalues $\lambda^2 - \lambda + \frac{1}{4} = 0 \Rightarrow \lambda_1 = \lambda_2 = \frac{1}{2} \Rightarrow |\lambda_{1,2}| < 1 \Rightarrow$
$(\bar{S}, \bar{I}) = (200, 50)$ stable

[Or Jury test: $Tr\, J = 1$, $\det J = \frac{1}{4} \Rightarrow \underbrace{|Tr\, J|}_{=1} < \underbrace{1 + \det J}_{=5/4} < 2$ satisfied $\Rightarrow$ stable]

(5) $u(t,x) = v(t,x)e^{\alpha t} \Rightarrow u_t = (v_t + \alpha v)e^{\alpha t},\ u_{xx} = v_{xx}e^{\alpha t} \Rightarrow$

$u_t = u_{xx} + u \Leftrightarrow (v_t + \alpha v)\cancel{e^{\alpha t}} = (v_{xx} + v)\cancel{e^{\alpha t}}$. Take $\alpha = 1 \Rightarrow v_t = v_{xx}$

IBVP for $v(t,x)$:

$$\begin{cases} v_t = v_{xx} \\ v(t,0) = u(t,0)e^{-t} = 0 \\ v(t,\pi) = u(t,\pi)e^{-t} = 0 \\ v(0,x) = u(0,x)e^{-0} = x \end{cases}$$

Separation of variables, $v(t,x) = T(t)\,X(x) \Rightarrow$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant} \Rightarrow T(t) = e^{\lambda t}$$

$v(t,0) = v(t,\pi) = 0 \Rightarrow X(0) = X(\pi) = 0$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases} \Rightarrow X_n(x) = \sin nx,\ n = 1, 2, \ldots,\ \lambda = -n^2 \Rightarrow$$

$$v(t,x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin nx \Rightarrow v(0,x) = \sum_{n=1}^{\infty} b_n \sin nx = x,$$ sin-series of x on $0 < x < \pi$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \underbrace{\frac{2}{\pi}\left[x\,\frac{-\cos nx}{n}\right]_0^{\pi}}_{=-\pi\frac{(-1)^n}{n} - 0} - \frac{2}{\pi}\int_0^{\pi} 1\cdot\frac{-\cos nx}{n}\,dx = \frac{2(-1)^{n+1}}{n} + \underbrace{\frac{2}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^{\pi}}_{=0} \Rightarrow$$

$$\underline{u(t,x)} = v(t,x)e^t = 2e^t\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2 t}\sin nx = \underline{2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{(1-n^2)t}\sin nx} =$$

$$= 2\Big(\sin x - \underbrace{e^{-3t}\frac{\sin 2x}{2} + e^{-8t}\frac{\sin 3x}{3} - e^{-15t}\frac{\sin 4x}{4} + \ldots}_{\to 0,\ t\to\infty}\Big) \to \underline{2\sin x},\ t \to \infty$$

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(6) Spatially uniform steady states: $\begin{cases} \bar N(6 - 3\bar N - \bar P) = 0 & (1) \\ \bar P(4 - \bar P - \bar N) = 0 & (2) \end{cases} \Rightarrow \bar P = 0$ or $\bar P = 4 - \bar N$

$\bar P = 0$ in (1) $\Rightarrow \bar N(6 - 3\bar N) = 0 \Rightarrow \bar N = 0$ or $\bar N = 2$

$\bar P = 4 - \bar N$ in (1) $\Rightarrow \bar N(2 - 2\bar N) = 0 \Rightarrow \bar N = 0$ or $\bar N = 1 \Rightarrow$

$\bar N = 0 \Downarrow \bar P = 4$, $\bar N = 1 \Downarrow \bar P = 3$

Spat. unif. steady states: $(0,0), (2,0), (0,4), (1,3)$

$$J(N,P) = \begin{pmatrix} 6 - 6N - P & -N \\ -P & 4 - 2P - N \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix};\ \begin{matrix}\lambda_1 = 6 > 0 \\ \lambda_2 = 4 > 0\end{matrix}\Big\}\text{unstable}$$

$$J(2,0) = \begin{pmatrix} -6 & -2 \\ 0 & 2 \end{pmatrix}\ \begin{matrix}\lambda_1 = -6 < 0 \\ \underline{\lambda_2 = 2 > 0}\end{matrix}\Big\}\text{unstable (saddle)},\quad J(0,4) = \begin{pmatrix} 2 & 0 \\ -4 & -4 \end{pmatrix}\ \begin{matrix}\lambda_1 = 2 > 0 \\ \lambda_2 = -4 < 0\end{matrix}\Big\}\text{unstable (saddle)}$$

$$J(1,3) = \begin{pmatrix} -3 & -1 \\ -3 & -3 \end{pmatrix} = A \quad \lambda_{1,2} = -3 \pm \sqrt{3} < 0 \Rightarrow \text{stable} \left[\text{or } \begin{matrix}\operatorname{Tr} A = -6 < 0 \\ \det A = 6 > 0\end{matrix}\Big\} \Rightarrow \text{stable}\right]$$

$\Rightarrow$ Only (1,3) stable, can check condition for diffusive instability at this point: $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \Leftrightarrow \underbrace{-3D_2 - 3D_1}_{<0} > \underbrace{2\sqrt{6D_1D_2}}_{>0}$

Impossible to satisfy with $D_1 > 0$ and $D_2 > 0 \Rightarrow$

no Turing diffusive instability possible.