Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2024-11-01 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The time evolution of a population $N(t)$ is described by the equation

$$\frac{dN}{dt} = r(N - 2\alpha)(\alpha - N)N$$

where $r > 0$ and $\alpha > 0$ are constants. Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$ for different initial values $N(0)$?

2. A population is divided into three age classes. The number of individuals at time n in the classes are a_n, b_n and c_n, where a_n is the youngest class and c_n the oldest. The survival rate from the youngest age class to the middle class is 75%, and from the middle class to the oldest 50%. The average number of births from individuals in the three classes are 0.25, 0.75 and 0.5, respectively. This gives the linear system

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 1/4 & 3/4 & 1/2 \\ 3/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} .$$

for $n = 0, 1, 2, \ldots$. What is the approximate age distribution of the population for large times n (that is, what fractions of the entire population are in the different age classes a_n, b_n and c_n)? Hint: Show first that $\lambda = 1$ is the eigenvalue of largest absolute value of the matrix.

3. Let $S(t)$, $I(t)$ and $R(t)$ be the numbers of susceptibles, infective, and removed, respectively, in a SIRS epidemic model. The total population N is constant, $N = 40$, and with $R(t) = N - S(t) - I(t)$ it is sufficient to study a two-dimensional system for $S(t)$ and $I(t)$. With certain choices of parameter values, the system is

$$\begin{cases} \dfrac{dS}{dt} = 20 - 0.5 \cdot S - 0.5 \cdot I - 0.01 \cdot SI \\ \dfrac{dI}{dt} = 0.01 \cdot SI - 0.5 \cdot I \end{cases}$$

Find all steady states of the system and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field). Note that $S \geq 0$, $I \geq 0$, and $S + I \leq 40$. What happens to the number of susceptibles and infective as $t \to \infty$? Give an interpretation of the result.

PLEASE TURN

4. A simplified time discrete chemostat model, with bacteria and nutrient concentrations N_n and C_n, respectively, at time n, is described by

$$\begin{cases} N_{n+1} = 2\,C_n N_n \\ C_{n+1} = -\,C_n N_n + 1 - \frac{1}{2} C_n \end{cases}$$

for $n = 0, 1, 2, \ldots$. Find all steady states and determine their stability.

5. For $0 < x < 1$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} 2u_t = u_{xx} \\ u(t,0) = 3 \\ u(t,1) = 5 \\ u(0,x) = 2x \end{cases}$$

Hint: put $u(t,x) = v(t,x) + Ax + B$ to get an IBVP with homogeneous boundary conditions for $v(t,x)$.

6. Let $f(x,t)$ be inhibitor concentration and $g(x,t)$ activator concentration in an activator-inhibitor model for pattern formation, which for two space dimensions is given by

$$\begin{cases} f_t = -f + \dfrac{1}{3} g^2 + D_1 \left(f_{xx} + f_{yy} \right) \\ g_t = \dfrac{1}{3} + \dfrac{g^2}{f} - \dfrac{10}{9} g + D_2 \left(g_{xx} + g_{yy} \right) \end{cases}$$

Find the spatially uniform steady state $(\bar{f}, \bar{g})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability.
Suppose now that $D_1 = 3$, $D_2 = 1/3$, that $0 < x < L_1 = \pi\sqrt{5/2}$, $0 < y < L_2 = \pi\sqrt{15}$, and that perturbations near $(\bar{f}, \bar{g})$ have the form $e^{\sigma t} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}$. For what values of (m, n) can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting two-dimensional patterns.

TATM38, 1/11, 2024, solution sketches

① $N' = f(N) = r(N-2\alpha)(\alpha - N)N$, steady states $f(\bar{N}) = 0 \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = \alpha, \bar{N}_3 = 2\alpha$

stable if $f'(\bar{N}_j) < 0$. $f'(N) = r(N-2\alpha)(\alpha-N) - r(N-2\alpha)N + r(\alpha-N)N \Rightarrow$

$f'(0) = -2r\alpha^2 < 0 \Rightarrow \bar{N}_1 = 0$ stable

$f'(\alpha) = r\alpha^2 > 0 \Rightarrow \bar{N}_2 = \alpha$ unstable

$f''(2\alpha) = -2r\alpha^2 < 0 \Rightarrow \bar{N}_3 = 2\alpha$ stable

$$N(t) \to \begin{cases} 2\alpha & \text{if } N(0) > \alpha \\ \alpha & \text{if } N(0) = \alpha \\ 0 & \text{if } N(0) < \alpha \end{cases}$$
as $t \to \infty$

② $\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = A \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$, $A = \begin{pmatrix} 1/4 & 3/4 & 1/2 \\ 3/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$, $\det(A - \lambda I) = -\lambda^3 + \frac{1}{4}\lambda^2 + \frac{9}{16}\lambda + \frac{3}{16} = 0$

$\lambda_1 = 1$ is a solution since $-1 + \frac{1}{4} + \frac{9}{16} + \frac{3}{16} = 0$. Polynomial division by $\lambda - 1$ gives the equation $\lambda^2 + \frac{3}{4}\lambda + \frac{3}{16} = 0 \Rightarrow \lambda_{2,3} = \frac{-3 \pm i\sqrt{3}}{8}$ with $|\lambda_{2,3}| = \frac{\sqrt{3}}{4} < 1$.

The general solution is $\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = C_1 \underbrace{\lambda_1^n}_{=1} \bar{v}_1 + C_2 \underbrace{\lambda_2^n}_{\to 0} \bar{v}_2 + C_3 \underbrace{\lambda_3^n}_{\to 0} \bar{v}_3 \to C_1 \bar{v}_1$, $n \to \infty$,

where $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are eigenvectors.

For large times, $\bar{v}_1$ gives the age distribution. $A\bar{v}_1 = 1 \cdot \bar{v}_1 \Rightarrow \bar{v}_1 = t \cdot \begin{pmatrix} 8 \\ 6 \\ 3 \end{pmatrix} =$

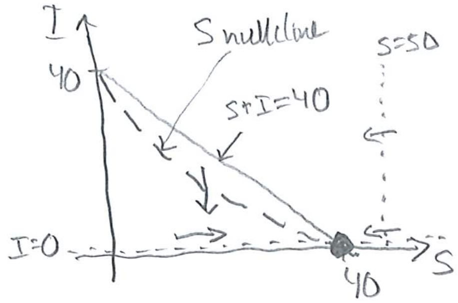
$= 17t \begin{pmatrix} 8/17 \\ 6/17 \\ 3/17 \end{pmatrix} \Rightarrow \frac{8}{17}$ in youngest class, $\frac{6}{17}$ in middle class, $\frac{3}{17}$ in oldest class, for large n

③ $\begin{cases} S' = 20 - 0.5S - 0.5I - 0.01SI \\ I' = 0.01SI - 0.5I \end{cases}$ nullcline $I(1 + \frac{S}{50}) = 40 - S \Rightarrow I = 50 \cdot \frac{40-S}{50+S}$ (dashed)

nullclines $I = 0$, $S = 50$ (dotted)

Steady states $I = 0 \Rightarrow S = 40$, $S = 50 \Rightarrow I = -5 < 0$ (and $S = 50 > 40 = N$ = total population)

$\Rightarrow (\bar{S}, \bar{I}) = (40, 0)$ is the only steady state with $S \geq 0, I \geq 0, S + I \leq 40$.



$$J(S,I) = \begin{pmatrix} -0.5 - 0.01I & -0.5 - 0.01S \\ 0.01I & 0.01S - 0.5 \end{pmatrix}$$

$$J(40,0) = \begin{pmatrix} -0.5 & -0.9 \\ 0 & -0.1 \end{pmatrix} \quad \left.\begin{matrix} \lambda_1 = -0.5 < 0 \\ \lambda_2 = -0.1 < 0 \end{matrix}\right\} \Rightarrow (40,0) \text{ stable}$$

As $t \to \infty$: $\begin{cases} S(t) \to 40 \ (= N) \\ I(t) \to 0 \end{cases}$

The disease will vanish from the population.

$\left(R_0 = \frac{0.01}{0.5} \cdot 40 = 0.8 < 1 \right)$

(4) $\begin{cases} N_{n+1} = 2C_nN_n \\ C_{n+1} = -C_nN_n + 1 - \frac{1}{2}C_n \end{cases}$ steady states $\begin{cases} \bar N = 2\bar C\bar N \\ \bar C = -\bar C\bar N + 1 - \frac{1}{2}\bar C \end{cases}$

$\Leftrightarrow \begin{cases} \bar N(1-2\bar C) = 0 & (1) \\ \bar C\bar N + \frac{3}{2}\bar C = 1 & (2) \end{cases}$ $(1) \Rightarrow \bar N = 0 \overset{(2)}{\Downarrow} \bar C = 2/3$ or $\bar C = 1/2 \overset{(2)}{\Downarrow} \bar N = 1/2$ $\Rightarrow$ Two steady states: $(0, 2/3)$ and $(1/2, 1/2)$

$J(N,C) = \begin{pmatrix} 2C & 2N \\ -C & -N-\frac{1}{2} \end{pmatrix} \Rightarrow J(0,\frac{2}{3}) = \begin{pmatrix} 4/3 & 0 \\ -2/3 & -1/2 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 4/3 > 1 \\ \lambda_2 = -1/2 \end{cases} \Rightarrow \underline{(0,\frac{2}{3}) \text{ unstable}}$

$J(1/2,1/2) = \begin{pmatrix} 1 & 1 \\ -1/2 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm\frac{1}{\sqrt{2}}, |\lambda_{1,2}| < 1 \Rightarrow \underline{(\frac{1}{2},\frac{1}{2}) \text{ stable}}$

[Jury test $\underbrace{|\mathrm{Tr}\, J|}_{0} < 1 + \underbrace{\det J}_{-1/2} < 2$ OK!]

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(5) $\begin{cases} 2u_t = u_{xx} \\ u(t,0) = 3 \\ u(t,1) = 5 \\ u(0,x) = 2x \end{cases}$ $u(t,x) = v(t,x) + Ax + B \Rightarrow u_t = v_t, u_{xx} = v_{xx}$

$u(t,0) = v(t,0) + B = 3$
$u(t,1) = v(t,1) + A + B = 5$ $\Big\} \Rightarrow$ with $B = 3$ and $A = 2$ one gets $v(t,0) = v(t,1) = 0$

Then $u(0,x) = v(0,x) + 2x + 3 = 2x \Rightarrow v(0,x) = -3$

$\Rightarrow \begin{cases} 2v_t = v_{xx} \\ v(t,0) = 0 \\ v(t,1) = 0 \\ v(0,x) = -3 \end{cases}$ $v(t,x) = T(t)X(x) \Rightarrow \frac{2T'}{T} = \frac{X''}{X} = \lambda = \text{constant}$

$v(t,0) = v(t,1) = 0 \Rightarrow X(0) = X(1) = 0$

$T' = \frac{\lambda}{2}T \Rightarrow T(t) = e^{\lambda t/2} \cdot \text{constant}$

$\begin{rcases} X'' - \lambda X = 0 \\ X(0) = X(1) = 0 \end{rcases} \Rightarrow X_n(x) = \sin(n\pi x), n = 1, 2, \dots, \lambda = -n^2\pi^2, T_n(t) = e^{-n^2\pi^2 t/2}$

$\Rightarrow v(t,x) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 t/2} \sin(n\pi x)$. $v(0,x) = \sum_1^\infty \alpha_n \sin(n\pi x) = -3 \Rightarrow$

$\alpha_n = 2\int_0^1 (-3)\sin n\pi x\, dx = \frac{6}{n\pi}\left[\cos n\pi x\right]_0^1 = \frac{-6}{n\pi}(1-(-1)^n) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-12}{n\pi} & \text{if } n \text{ odd} \end{cases} \Rightarrow$

$\underline{u(t,x)} = 2x + 3 + v(t,x) = \underline{2x + 3 - \frac{6}{\pi}\sum_1^\infty \frac{1-(-1)^n}{n} e^{-n^2\pi^2 t/2} \sin(n\pi x)}$

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(6) Spatially uniform steady state $\begin{cases} -\bar f + \frac{1}{3}\bar g^2 = 0 \\ \frac{1}{3} + \bar g^2/\bar f - \frac{10}{9}\bar g = 0 \end{cases}$ $\Rightarrow \bar g^2/\bar f = 3 \Rightarrow \frac{10}{3} = \frac{10}{9}\bar g \Rightarrow \bar g = 3 \Rightarrow \bar f = 3 \Rightarrow (\bar f, \bar g) = (3,3)$

$J(f,g) = \begin{pmatrix} -1 & 2g/3 \\ -g^2/f^2 & 2g/f - 10/9 \end{pmatrix} \Rightarrow J(3,3) = \begin{pmatrix} -1 & 2 \\ -1 & 8/9 \end{pmatrix} = A$ $\begin{rcases} \mathrm{Tr}\, A = -\frac{1}{9} < 0 \\ \det A = \frac{10}{9} > 0 \end{rcases} \Rightarrow (3,3)$ stable without diffusion

Turing condition $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1D_2 \det A} \Rightarrow -D_2 + \frac{8}{9}D_1 > \frac{2}{3}\sqrt{10D_1D_2}$

Check $D_1 = 3, D_2 = 1/3$: $-\frac{1}{3} + \frac{8}{3} > \frac{2}{3}\sqrt{10} \Leftrightarrow 7 > 2\sqrt{10}$ OK (since $49 > 4 \cdot 10$)

Unstable perturbations if

$0 > \det(A - Q^2D) = \begin{vmatrix} -1-3Q^2 & 2 \\ -1 & \frac{8}{9} - \frac{Q^2}{3} \end{vmatrix} = Q^4 - \frac{7}{3}Q^2 + \frac{10}{9} = (Q^2 - \frac{7}{6})^2 - \frac{1}{4} \Leftrightarrow |Q^2 - \frac{7}{6}| < \frac{1}{2} \Leftrightarrow$

$-\frac{1}{2} < Q^2 - \frac{7}{6} < \frac{1}{2} \Leftrightarrow \frac{2}{3} < Q^2 < \frac{5}{3}$. With $Q^2 = (\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2})\pi^2 = \frac{2m^2}{5} + \frac{n^2}{15}$, $\underline{10 < 6m^2 + n^2 < 25}$

Satisfied for $(m,n) = \underline{(0,4), (1,3), (1,4), (2,0)}$ Patterns: