Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

**Written examination, TATM38 Mathematical Models in Biology**

**2025-01-09 , 14.00 - 19.00**

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

**1.** (a) The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0.25 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Determine the general solution to this system and draw a picture of the entire phase plane (including negative $x$ and $y$). Is the steady state (= equilibrium point) $(0,0)$ stable?

(b) Give also the general solution to the system of difference equations

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0.25 & -1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} .$$

Is $(0,0)$ stable in this case?

**2.** A population $N_n$ is described by the time discrete model

$$N_{n+1} = \frac{3N_n}{1+N_n} \quad , \quad n = 0, 1, 2, \ldots$$

Find all steady states (= equilibrium points) and determine their stability. What happens to the population as $n \to \infty$ if $N_0 > 0$? Sketch a cobweb diagram.

**3.** An ecosystem model for phytoplankton $P(t)$, zooplankton $Z(t)$, and nitrogen leads to the two-dimensional dynamical system for $P(t)$ and $Z(t)$:

$$\begin{cases} \dfrac{dP}{dt} = P(2 - 2P - Z) \\ \dfrac{dZ}{dt} = Z(2P - 1) \end{cases}$$

Find all steady states of the system and determine their stability. Draw, for $P \geq 0$ and $Z \geq 0$, a phase plane picture with nullclines and directions of the vector field. What happens to the populations as $t \to \infty$ if $P(0) > 0$ and $Z(0) > 0$?

**PLEASE TURN**

**4.** A model for two populations $x(t)$ and $y(t)$ is given by

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \dfrac{2x}{1+y}\right) \\[2ex] \dfrac{dy}{dt} = y\left(1 - \dfrac{y}{2+x}\right) \end{cases}$$

Find all steady states and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field).
What happens to the populations as $t \to \infty$ if a) $x(0) = 0$ and $y(0) > 0$, b) $x(0) > 0$ and $y(0) = 0$, c) $x(0) > 0$ and $y(0) > 0$?
Explain why this a mutualist population model. (Mutualist means that both populations benefit from the presence of the other population.)

**5.** For $0 < x < 2$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} 2u_t = u_{xx} - 2u \\ u_x(t, 0) = 0 \\ u_x(t, 2) = 0 \\ u(0, x) = 3 + 4\cos \pi x \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

**6.** Consider a reactor-diffusion system in one space dimension for the functions $u(t, x)$ and $v(t, x)$

$$\begin{cases} u_t = R_1(u, v) + D_1\, u_{xx} \\ v_t = R_2(u, v) + D_2\, v_{xx} \end{cases}$$

In a spatially uniform steady state $(\bar{u}, \bar{v})$ (this means $R_1(\bar{u}, \bar{v}) = R_2(\bar{u}, \bar{v}) = 0$), for which of the following three Jacobi matrices $J(\bar{u}, \bar{v})$ can we have Turing diffusive instabilities?

$$J_1 = \begin{pmatrix} -1 & 3 \\ -1 & -1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \qquad J_3 = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$$

Justify your answer clearly.

For the matrix that allows diffusive instability, assume now that $0 < x < 6\pi$, $D_1 = 9$ and $D_2 = 1$. Find the values of $n$ for which unstable (pattern forming) modes $e^{\sigma t} \cos \frac{nx}{6}$ $(\sigma > 0)$ appear, and sketch the resulting one-dimensional patterns.

# TATM38, Math. Bio., 9/1 2025, solution sketches

① a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ 0.25 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$, Eigenvalues $\begin{vmatrix} -1-\lambda & 1 \\ 0.25 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 0.75 = 0$

$\Rightarrow \lambda_1 = -\frac{1}{2}$, $\lambda_2 = -\frac{3}{2}$, Eigenvectors $\lambda_1 = -\frac{1}{2} \Rightarrow \bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\lambda_2 = -\frac{3}{2} \Rightarrow \bar{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

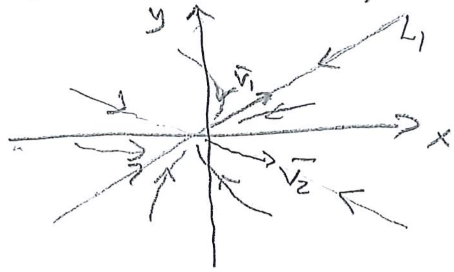
General solution is $\begin{pmatrix} x \\ y \end{pmatrix} = k_1 e^{\lambda_1 t} \bar{v}_1 + k_2 e^{\lambda_2 t} \bar{v}_2 = k_1 e^{-t/2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{-3t/2} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$k_1, k_2$ constants

phase plane:

$(0,0)$ is stable since both $\lambda_1 < 0$ and $\lambda_2 < 0$

(solutions approach $(0,0)$ along $L_1$ as $t \to \infty$)




b) $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ has general solution

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 \lambda_1^n \bar{v}_1 + c_2 \lambda_2^n \bar{v}_2 = c_1 \left(-\tfrac{1}{2}\right)^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(-\tfrac{3}{2}\right)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad c_1, c_2 \text{ constants}$$

$(0,0)$ is <u>unstable</u> since $|\lambda_2| = \frac{3}{2} > 1$

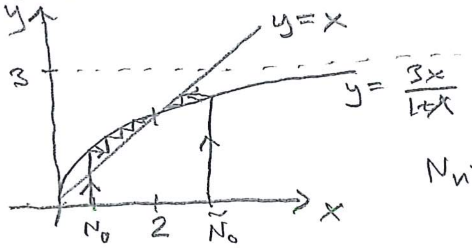
---

② $\bar{N}$ steady state of $N_{n+1} = \dfrac{3N_n}{1+N_n}$ if $\bar{N} = \dfrac{3\bar{N}}{1+\bar{N}} \Rightarrow \bar{N}_1 = 0$ or $\bar{N}_2 = 2$

Stability $f(N) = \dfrac{3N}{1+N} \Rightarrow f'(N) = \dfrac{3}{(1+N)^2} \Rightarrow |f'(0)| = 3 > 1$ and $|f'(2)| = \frac{1}{3} < 1$

$\Rightarrow \begin{cases} \bar{N}_1 = 0 \text{ unstable} \\ \bar{N}_2 = 2 \text{ stable} \end{cases}$

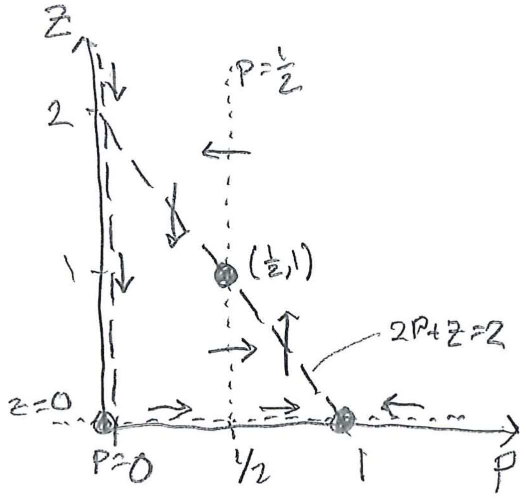
Cobweb




$N_n \to 2$ as $n \to \infty$ (if $N_0 > 0$)

---

③ $\begin{cases} P' = P(2 - 2P - Z) \\ Z' = Z(2P - 1) \end{cases}$ P nullclines: $P = 0$, $2P + Z = 2$ – – –
Z nullclines: $Z = 0$, $P = 1/2$ ........




3 steady states: $(\bar{P}_1, \bar{Z}_1) = (0,0)$, $(\bar{P}_2, \bar{Z}_2) = (1,0)$, $(\bar{P}_3, \bar{Z}_3) = (\frac{1}{2}, 1)$

$J(P,Z) = \begin{pmatrix} 2-4P-Z & -P \\ 2Z & 2P-1 \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{cases} \lambda_1 = 2 > 0 \\ \lambda_2 = -1 < 0 \end{cases}$

$\Rightarrow$ saddle, unstable

$J(1,0) = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \begin{cases} \lambda_1 = -2 < 0 \\ \lambda_2 = 1 > 0 \end{cases} \Rightarrow$ saddle, unstable

$J(\frac{1}{2}, 1) = \begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix} = J$ $\left.\begin{matrix} \text{Tr} J = -1 < 0 \\ \det J = 1 > 0 \end{matrix}\right\} \Rightarrow$ stable

[or $\det(J - \lambda I) = \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \Rightarrow$
$Re(\lambda_{1,2}) = -\frac{1}{2} < 0 \Rightarrow$ stable (spiral)]

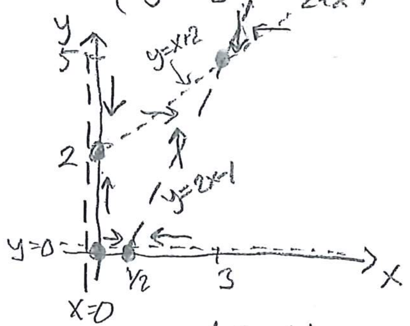
$(P(t), Z(t)) \to (\frac{1}{2}, 1)$ as $t \to \infty$ if $P(0) > 0$ and $Z(0) > 0$

Populations can coexist.

---

(4) $\begin{cases} x' = 2x\left(1-\frac{2x}{1+y}\right) \\ y' = y\left(1-\frac{y}{2+x}\right) \end{cases}$ x nullclines $x=0,\ 2x=1+y \Leftrightarrow y=2x-1$

y nullclines $y=0,\ y=2+x$




4 steady states $(0,0), (\frac{1}{2},0), (0,2)$ and $(3,5)$ $\left(y=2+x=2x-1 \Rightarrow x=3 \Rightarrow y=5\right)$

$$J(x,y) = \begin{pmatrix} 2-\frac{8x}{1+y} & \frac{4x^2}{(1+y)^2} \\ \frac{y^2}{(2+x)^2} & 1-\frac{2y}{2+x} \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \quad \begin{matrix} \lambda_1 = 2>0 \\ \lambda_2 = 1>0 \end{matrix} \Big\} \Rightarrow \text{unstable}$$

$$J(\tfrac{1}{2},0) = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \begin{matrix} \lambda_1=-2<0 \\ \lambda_2 = 1>0 \end{matrix} \Big\} \Rightarrow \text{unstable (saddle)},\ J(0,2) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{matrix} \lambda_1 = 2>0 \\ \lambda_2=-1<0 \end{matrix} \Big\} \Rightarrow \text{unstable (saddle)}$$

$$J(3,5) = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}.\ \lambda_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2} < 0 \Rightarrow \text{stable (or } \det J = 1>0,\ \mathrm{Tr}\, J = -3<0 \Rightarrow \text{stable)}$$

a) $x(0)=0, y(0)>0 \Rightarrow (x(t),y(t)) \to (0,2)$ as $t\to\infty$ (along y-axis)

b) $x(0)>0, y(0)=0 \Rightarrow (x(t),y(t)) \to (\frac{1}{2},0)$ as $t\to\infty$ (along x-axis)

c) $x(0)>0, y(0)>0 \Rightarrow (x(t),y(t)) \to (3,5)$ as $t\to\infty$, larger asymptotic values for both populations if both present (so mutualist) $(3>\frac{1}{2},\ 5>2)$

---

(5) $u(t,x)=v(t,x)e^{\alpha t} \Rightarrow u_t=(v_t+\alpha v)e^{\alpha t},\ u_x=v_x e^{\alpha t},\ u_{xx}=v_{xx}e^{\alpha t} \Rightarrow$

$2u_t = u_{xx}-2u \Leftrightarrow 2(v_t+\alpha v)\cancel{e^{\alpha t}} = (v_{xx}-2v)\cancel{e^{\alpha t}}$. Choose $\alpha=-1$ to get $2v_t=v_{xx}$

and the IBVP for $v(t,x)$:

$$\begin{cases} 2v_t = v_{xx} \\ v_x(t,0) = u_x(t,0)e^t = 0 \\ v_x(t,2) = u_x(t,2)e^t = 0 \\ v(0,x) = u(0,x)e^0 = 3+4\cos\pi x \end{cases}$$

Separation of variables $v(t,x)=T(t)X(x) \Rightarrow$

$$\frac{2T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant} \Rightarrow T(t)=e^{\lambda t/2}$$

$$\begin{matrix} v_x(t,0) = T(t)X'(0)=0 \\ v_x(t,2) = T(t)X'(2)=0 \end{matrix} \Big\} \Rightarrow X'(0)=X'(2)=0$$

$$\begin{matrix} X''(x)-\lambda X(x)=0 \\ X'(0)=X'(2)=0 \end{matrix} \Big\} \Rightarrow X_n(x) = \cos\frac{n\pi x}{2},\ n=0,1,2,\dots \text{ (non-zero solutions)},\ \lambda=-\frac{n^2\pi^2}{4}$$

$\Rightarrow T_n(t)=e^{-n^2\pi^2 t/8}$

Linear PDE and homog. boundary conditions $\Rightarrow$

$v(t,x) = \sum_{n=0}^{\infty} \alpha_n e^{-n^2\pi^2 t/8}\cos\frac{n\pi x}{2}$. Initial condition $v(0,x)=3+4\cos\pi x$ $(t=0)$ $\Rightarrow$

$\sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{2} = 3+4\cos\pi x$, identify $\alpha_0=3, \alpha_2=4, \alpha_n=0$ for $n\neq 0,2$ $\Rightarrow$

$$v(t,x) = 3+4e^{-\pi^2 t/2}\cos\pi x \Rightarrow u(t,x) = 3e^{-t}+4e^{-(1+\frac{\pi^2}{2})t}\cos\pi x$$

---

(6) The Turing conditions for diffusive instabilities are

1. $\mathrm{Tr}\, J<0$
2. $\det J>0$
3. $j_{11}D_2+j_{22}D_1 > 2\sqrt{D_1D_2\det J}$

$J_1 = \begin{pmatrix} -1 & 3 \\ -1 & -1 \end{pmatrix}$ has $j_{11}=j_{22}=-1 \Rightarrow$ 3. cannot be satisfied

$J_2 = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$ has $\mathrm{Tr}\, J_2 = -1+2=1>0 \Rightarrow$ 1. not satisfied

$J_3 = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ has $\mathrm{Tr}\, J_3=-1<0$ (1. satisfied), $\det J_3 = 1$ (2. satisfied) and 3. is $-2D_2+D_1 > 2\sqrt{D_1D_2}$ which can be satisfied (depending on $D_1$ and $D_2$). With $D_1=9$ and $D_2=1$, $-2D_2+D_1=-2+9=7>$ $>6=2\sqrt{9\cdot 1}$ so 3. satisfied.

With $\mu = \frac{j_{11}D_2+j_{22}D_1}{D_1D_2} = \frac{7}{9}$, $w = \frac{\det J}{D_1D_2} = \frac{1}{9}$, $\Delta = \sqrt{\frac{\mu^2}{4}-w} = \frac{\sqrt{13}}{18}$, unstable perturbations appear for $\frac{\mu}{2}-\Delta < q^2 = \frac{n^2}{36} < \frac{\mu}{2}+\Delta \Leftrightarrow 14-2\sqrt{13} < n^2 < 14+2\sqrt{13}$ $\left(3<\sqrt{13}<4 \Rightarrow \begin{matrix} 6<14-2\sqrt{13}<8 \\ 20<14+2\sqrt{13}<22 \end{matrix}\right)$

$\Rightarrow n=3$ or $n=4$ ;