Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2025-10-31 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The time evolution of a population $N(t)$ is described by the equation

$$\frac{dN}{dt} = r(1 - e^{N-\alpha})N$$

where $r > 0$ and $\alpha > 0$ are constants. Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$ for different initial values $N(0) \geq 0$?

2. Consider the time discrete model

$$x_{n+1} = \frac{9x_n}{(2+x_n)^2} \quad , \quad n = 0, 1, 2, \ldots$$

Find the two non-negative steady states (= equilibrium points) and determine their stability. What happens to x_n as $n \to \infty$ if $x_0 > 0$? Sketch a cobweb diagram (for the positive x-axis) for some $x_0 > 0$.

3. A model for two interacting animal populations $x(t)$ and $y(t)$ is given by

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{2}\right) - xy \\ \dfrac{dy}{dt} = y\left(1 - \dfrac{y}{1+x}\right) \end{cases}$$

Find all steady states and determine their stability. Draw, for $x \geq 0$ and $y \geq 0$, a phase plane picture (with nullclines and directions of the vector field). What happens to the populations as $t \to \infty$ if $x(0) > 0$ and $y(0) > 0$?

PLEASE TURN

4. Let S_n, I_n and R_n be the number of susceptibles, infective, and removed, respectively, at time $n = 0, 1, 2, 3, \ldots$ in a time discrete SIRS epidemic model. The total population N is constant, $N = 200$, and with $R_n = N - S_n - I_n$ one has a two-dimensional system for S_n and I_n:

$$\begin{cases} S_{n+1} = S_n - \dfrac{S_n I_n}{100} + \dfrac{1}{4}(200 - S_n - I_n) \\[2ex] I_{n+1} = I_n + \dfrac{S_n I_n}{100} - \dfrac{I_n}{2} \end{cases}$$

Find all steady states (= equilibrium points) and determine their stability.

5. For $0 < x < 2$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} u_t = 2u_{xx} - u \\ u_x(t, 0) = 0 \\ u_x(t, 2) = 0 \\ u(0, x) = 5 - \cos \pi x \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

6. Consider a two-component reaction-diffusion system in two space dimensions with concentrations $u(t, x, y)$ and $v(t, x, y)$:

$$\begin{cases} u_t = 0.25 - uv^2 + D_1 (u_{xx} + u_{yy}) \\ v_t = 0.25 + 3uv^2 - v + D_2 (v_{xx} + v_{yy}) \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = \sqrt{20}$, $D_2 = 1/\sqrt{20}$, that $0 < x < L_1 = \pi\sqrt{14\sqrt{5}}$, $0 < y < L_2 = \pi\sqrt{7\sqrt{5}}$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}$. For what values of (m, n) can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting two-dimensional patterns.

TATM38, 31/10 2025, solution sketches

① $N'=f(N) = r(1-e^{N-\alpha})N$, steady states $f(\bar{N})=0 \Rightarrow \bar{N}_1=0, \bar{N}_2=\alpha$

Stable if $f'(\bar{N}_j)<0$. $f'(N) = r(1-e^{N-\alpha}) - re^{N-\alpha}N \Rightarrow$

$f'(0) = r(1-e^{-\alpha})+0 > 0$ (since $\alpha>0$) $\Rightarrow \bar{N}_1=0$ unstable

$f'(\alpha) = r(1-1) - r\cdot 1\cdot\alpha = -r\alpha<0 \Rightarrow \bar{N}_2=\alpha$ stable

$N(t) \to \begin{cases} \alpha \text{ if } N(0)>0 \\ 0 \text{ if } N(0)=0 \end{cases}$

as $t\to\infty$

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② $x_{n+1} = f(x_n) = \dfrac{9x_n}{(2+x_n)^2}$. Steady states $\bar{x} = f(\bar{x}) = \dfrac{9\bar{x}}{(2+\bar{x})^2} \Rightarrow \underline{\bar{x}_1=0}$ or

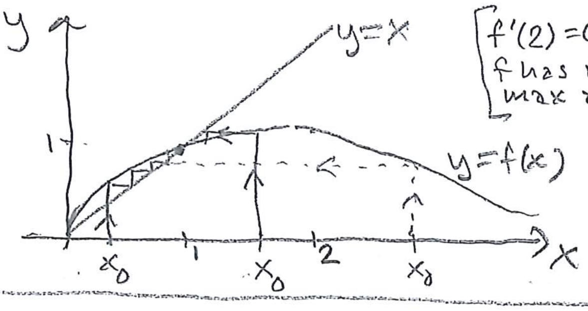
$9=(2+\bar{x})^2 \Rightarrow 2+\bar{x}=\pm 3 \Rightarrow \underline{\bar{x}_2=1}$ (and $\bar{x}_3=-5<0$, excluded).

stable if $|f'(\bar{x}_j)|<1$. $f'(x) = \dfrac{9(2+x)^2-9x\cdot 2(2+x)}{(2+x)^4} = \dfrac{9(2-x)}{(2+x)^3} \Rightarrow$

$|f'(0)| = \dfrac{18}{8} > 1 \Rightarrow \bar{x}_1=0$ unstable

$|f'(1)| = \dfrac{9}{27} = \dfrac{1}{3}<1 \Rightarrow \bar{x}_2=1$ stable

If $x_0>0$, then $\underline{x_n \to 1 \text{ as } n\to\infty}$



[$f'(2)=0 \Rightarrow$ f has its max at $x=2$]

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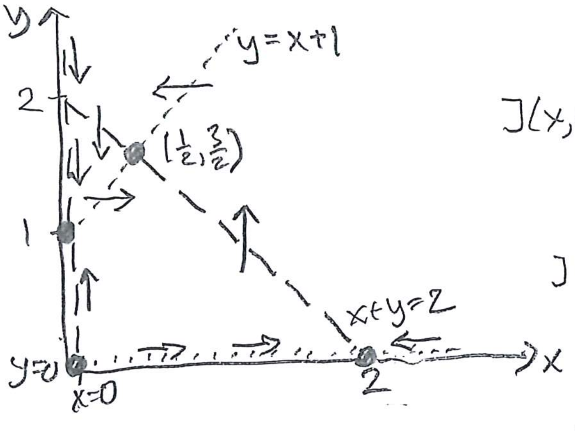
③ $\begin{cases} x' = 2x(1-\frac{x}{2}) - xy = x(2-x-y) \\ y' = y(1-\frac{y}{1+x}) \end{cases}$, x nullclines $x=0, x+y=2$ 2 lines

, y nullclines $y=0, y=1+x$ 2 lines

Phase plane (for $x\geq 0, y\geq 0$)



4 steady states: $(\bar{x}_1,\bar{y}_1)=(0,0), (\bar{x}_2,\bar{y}_2)=(2,0)$
$(\bar{x}_3,\bar{y}_3)=(0,1), (\bar{x}_4,\bar{y}_4)=(\frac{1}{2},\frac{3}{2})$

$J(x,y) = \begin{pmatrix} 2-2x-y & -x \\ \frac{y^2}{(1+x)^2} & 1-\frac{2y}{1+x} \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{cases} \lambda_1=2>0 \\ \lambda_2=1>0 \end{cases}$

$\Rightarrow$ unstable

$J(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix} \begin{cases} \lambda_1=-2<0 \\ \lambda_2=1>0 \end{cases}$, $J(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{cases} \lambda_1=1>0 \\ \lambda_2=-1<0 \end{cases}$

saddle, unstable   saddle, unstable

$J(\frac{1}{2},\frac{3}{2}) = \begin{pmatrix} -1/2 & -1/2 \\ 1 & -1 \end{pmatrix} = J$ $\left.\begin{matrix} \text{Tr } J = -\frac{3}{2}<0 \\ \det J = 1 >0 \end{matrix}\right\} \Rightarrow$ stable $\left[\lambda_{1,2} = \underbrace{-\frac{3}{4}}_{<0} \pm i\frac{\sqrt{7}}{4} \notin \mathbb{R} \Rightarrow \text{spiral}\right]$

The populations approach the steady state $(\bar{x}_4,\bar{y}_4)=(\frac{1}{2},\frac{3}{2})$ as $t\to\infty$

④ $\begin{cases} S_{n+1} = S_n - \dfrac{S_n I_n}{100} + \frac{1}{4}(200 - S_n - I_n) \\ I_{n+1} = I_n + \dfrac{S_n I_n}{100} - \dfrac{I_n}{2} \end{cases}$ Steady states $\begin{cases} \bar{S} = \bar{S} - \dfrac{\bar{S}\bar{I}}{100} + \frac{1}{4}(200 - \bar{S} - \bar{I}) \quad \circledast \\ \bar{I} = \bar{I} + \dfrac{\bar{S}\bar{I}}{100} - \dfrac{\bar{I}}{2} \Rightarrow \bar{I} = 0 \text{ or } \bar{S} = 50 \end{cases}$

$\bar{I} = 0$ in $\circledast \Rightarrow \bar{S} = 200$, $\bar{S} = 50$ in $\circledast \Rightarrow -\frac{\bar{I}}{2} + \frac{1}{4}(150 - \bar{I}) = 0 \Rightarrow \bar{I} = 50$

Two steady states $(\bar{S}_1, \bar{I}_1) = (200, 0)$, $(\bar{S}_2, \bar{I}_2) = (50, 50)$ (with $\bar{R}_1 = 0$ and $\bar{R}_2 = 100$, resp.)

$$J(S,I) = \begin{pmatrix} 1 - \frac{I}{100} - \frac{1}{4} & -\frac{S}{100} - \frac{1}{4} \\ \frac{I}{100} & 1 + \frac{S}{100} - \frac{1}{2} \end{pmatrix} \Rightarrow J(200,0) = \begin{pmatrix} 3/4 & -9/4 \\ 0 & 5/2 \end{pmatrix} \quad \left.\begin{matrix} \lambda_1 = \frac{3}{4} \\ \lambda_2 = \frac{5}{2} \end{matrix}\right\} |\lambda_2| > 1 \Rightarrow \underline{\text{unstable}}$$

$$J(50,50) = \begin{pmatrix} 1/4 & -3/4 \\ 1/2 & 1 \end{pmatrix} = J \quad \begin{matrix} \operatorname{Tr} J = \frac{5}{4} \\ \det J = \frac{5}{8} \end{matrix} \Rightarrow \underbrace{|\operatorname{Tr} J|}_{5/4} < \underbrace{1 + \det J}_{13/8} < 2$$ satisfied, so $\underline{\text{stable}}$ by Jury test

[or $|\lambda_{1,2}| = \left|\frac{5 \pm i\sqrt{15}}{8}\right| = \frac{1}{8}\sqrt{25+15} = \frac{\sqrt{40}}{8} = \sqrt{\frac{5}{8}} < 1 \Rightarrow$ stable]

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⑤ $u(t,x) = v(t,x)e^{\alpha t} \Rightarrow u_x = v_x e^{\alpha t}$, $u_{xx} = v_{xx}e^{\alpha t}$, $u_t = (v_t + \alpha v)e^{\alpha x}$

$u_t = 2u_{xx} - u \Leftrightarrow v_t + \alpha v = 2v_{xx} - v$. Take $\underline{\alpha = -1} \Rightarrow u = ve^{-t}$, $v = ue^{t}$, $v_t = 2v_{xx}$

IBVP for $v(t,x)$

$$\begin{cases} v_t = 2v_{xx} \\ v_x(t,0) = u_x(t,0)e^t = 0 \\ v_x(t,2) = u_x(t,2)e^t = 0 \end{cases} \Big\} \text{BC} \\ v(0,x) = u(0,x) \cdot e^0 = 5 - \cos \pi x$$

$v(t,x) = T(t)X(x) \Rightarrow \dfrac{T'(t)}{2T(t)} = \dfrac{X''(x)}{X(x)} = \lambda$

$X'(0) = X'(2) = 0$ from BC

gives $X_n(x) = \cos\frac{n\pi x}{2}$, $n = 0, 1, 2, \ldots$, $\lambda = -\frac{n^2\pi^2}{4}$

and $T_n(t) = e^{2\lambda t} = e^{-n^2\pi^2 t/2}$

$\Rightarrow v(t,x) = \sum_{n=0}^{\infty} \alpha_n e^{-n^2\pi^2 t/2} \cos\frac{n\pi x}{2}$ solves $v_t = 2v_{xx}$ + BC . Initial condition

$v(0,x) = 5 - \cos \pi x \Rightarrow \alpha_0 = 5$, $\alpha_2 = -1$ and $\alpha_n = 0$ for $n \neq 0, 2$ $\Rightarrow$

$\underline{u(t,x)} = e^{-t} v(t,x) = \underline{e^{-t}(5 - e^{-2\pi^2 t}\cos \pi x)}$

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⑥ Spatially uniform steady state $\left.\begin{matrix} 0{,}25 - \bar{u}\bar{v}^2 = 0 \\ 0{,}25 + 3\bar{u}\bar{v}^2 - \bar{v} = 0 \end{matrix}\right\} \Rightarrow (\bar{u}, \bar{v}) = (0{,}25, 1)$

$$J(u,v) = \begin{pmatrix} -v^2 & -2uv \\ 3v^2 & 6uv - 1 \end{pmatrix} \Rightarrow J(0{,}25, 1) = \begin{pmatrix} -1 & -0{,}5 \\ 3 & 0{,}5 \end{pmatrix} = A \quad \left.\begin{matrix} \operatorname{Tr} A = -0{,}5 < 0 \\ \det A = 1 > 0 \end{matrix}\right\} \Rightarrow (0{,}25, 1)$$ stable without diffusion

Turing condition: $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \Leftrightarrow \underline{-D_2 + 0{,}5D_1 > 2\sqrt{D_1 D_2}}$

$D_1 = \sqrt{20}$, $D_2 = \frac{1}{\sqrt{20}}$ gives $\sqrt{5} - \frac{1}{\sqrt{20}} > 2$ which is OK since $\sqrt{5} - \frac{1}{\sqrt{20}} = \frac{\sqrt{100} - 1}{\sqrt{20}} = \frac{9}{\sqrt{20}} > 2$ since $\sqrt{20} < 4{,}5$

Unstable perturbations if

$$0 > \begin{vmatrix} -1 - \sqrt{20}Q^2 & -0{,}5 \\ 3 & 0.5 - \frac{Q^2}{\sqrt{20}} \end{vmatrix} = Q^4 - \frac{9}{\sqrt{20}}Q^2 + 1 = \left(Q^2 - \frac{9}{2\sqrt{20}}\right)^2 + 1 - \frac{81}{80} \Leftrightarrow \frac{-1}{\sqrt{80}} < Q^2 - \frac{9}{\sqrt{80}} < \frac{1}{\sqrt{80}} \Leftrightarrow$$

$\frac{2}{\sqrt{5}} < Q^2 < \frac{\sqrt{5}}{2}$ With $Q^2 = \left(\frac{m^2}{L_1^2} + \frac{n^2}{L_2^2}\right)\pi^2 = \frac{m^2}{14\sqrt{5}} + \frac{2n^2}{14\sqrt{5}}$ one gets $\underline{28 < m^2 + 2n^2 < 35}$

With solutions $(m,n) = \underline{(5,2), (4,3), (0,4), (1,4)}$ with patterns