A Brief History of Functional Analysis

Functional analysis was born in the early years of the twentieth century as part of a larger trend toward abstraction—what some authors have called the "arithmetization" of analysis. This same trend toward "axiomatics" contributed to the foundations of abstract linear algebra, modern geometry, and topology. Functional analysis is now a very broad field, encompassing much of modern analysis. In fact, it would be difficult to give a simple definition of what functional analysis means today. Rather than discuss its current meaning, we will concentrate on its foundations and settle for an all too brief description of modern trends.

We will discuss several episodes from the early history of "abstract analysis," especially those related to the development of vector spaces and other "abstract spaces." Of particular interest to us will be the movement from the specific to the generic in mathematics; as one example of this, you may be surprised to learn that the practice of referring to functions by "name," writing a single letter f, say, rather than the referring to its values $f(x)$, only became common in our own century.

In particular, we will discuss the work of Fredholm and Hilbert on integral equations and operator theory, the work of Volterra and Hadamard on the problem of moments, the work of Lebesgue, Fréchet, and Riesz on abstract spaces, and the work of Helly, Hahn, and Banach on the notion of duality. In addition, we will present a few examples which illustrate the functional analytic viewpoint.

1. Brief Summary of Important Dates

— Fredholm's 1900 paper on integral equations

— Lebesgue's 1902 thesis on integration

— Hilbert's paper of 1906 on spectral theory

— Fréchet's 1906 thesis on metric spaces

— Riesz's papers of 1910 and 1911 on $C[\,a,b\,]$ and L_p

— Helly's papers of 1912 and 1921

— Banach's thesis of 1922 on normed spaces

— Hahn's 1927 paper and Banach's 1929 paper on duality; the 1927 paper of Banach and Steinhaus (featuring Saks' proof)

— Fréchet's 1928 book *Les espaces abstrait*, and Banach's 1932 book *Théorie des opérations linéaires*

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2. Early Examples

Our first example of an "integral equation" will look familiar to our students of probability: In his famous book, *The Analytic Theory of Heat*, 1822, Fourier discussed the problem of "inverting" the equation

$$f(x) = \int_{-\infty}^{\infty} e^{itx} g(t)\, dt.$$

That is, we suppose that f is known and we seek a solution g to the integral equation. In modern language, we would say that f is the Fourier transform (or characteristic function) of g, and here we seek the inverse transform. Fourier offered the solution

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t)\, dt,$$

which is now known as the Fourier inversion formula.

As a second example, the Norwegian mathematician Niels Abel, in 1823, offered a solution to the so-called tautochrone problem in the form of an integral equation. In modern language, we suppose that we're given a curve $y = f(x)$ with $f(a) = 0$ and we seek a solution to the equation

$$f(x) = \int_a^x \frac{g(y)}{\sqrt{x-y}}\, dy,$$

where $a \le x \le b$, for which Abel offered the solution

$$g(x) = \frac{1}{\pi} \int_a^x \frac{f'(y)}{\sqrt{x-y}}\, dy.$$

While Abel's equation would be the subject of many later studies, his own contributions were not terribly influential.

By way of a final example: Liouville, in his research on second order linear differential equations in 1837, discovered that such equations could be written as integral equations. Consider, for example, the equation

$$f''(x) + f(x) = g(x)$$

with the initial conditions

$$f(a) = 1, \quad f'(a) = 0.$$

Now the solution to the homogenous equation can be written

$$f(x) = A\cos(x-a) + B\sin(x-a),$$

where A and B are constants. Direct substitution (or a few minutes spent applying variation of parameters) will convince you that a solution to the nonhomogeneous equation is

$$f(x) = \cos(x-a) + \int_a^x \sin(x-y)\, g(y)\, dy,$$

and so, in the special case where $g(x) = \sigma(x)f(x)$, we get

$$f(x) = \cos(x-a) + \int_a^x \sin(x-y)\, \sigma(y)\, f(y)\, dy.$$

By the middle of the nineteenth century, interest in integral equations centered around the solution of certain boundary value problems involving Laplace's equation

$$\triangle u = u_{xx} + u_{yy} = 0,$$

which were known to be equivalent to integral equations.

As we'll see, the study of integral equations is closely related to the study of systems of linear equations in infinitely many unknowns. This particular direction proved to be influential for only a short time, though. Perhaps a single example would tide us over for the moment: Consider the following system of equations offered up by Helly in 1921.

$$\begin{array}{ccccccccccc}
1 & = & x_1 & + & x_2 & + & x_3 & + & x_4 & + & \cdots \\
1 & = & & & x_2 & + & x_3 & + & x_4 & + & \cdots \\
1 & = & & & & & x_3 & + & x_4 & + & \cdots \\
1 & = & & & & & & & x_4 & + & \cdots \\
 & \vdots & & & & & & & & &
\end{array}$$

Clearly, if we "truncate" the system to any $n \times n$ square, we have the unique solution $x_1 = \cdots = x_{n-1} = 0$, $x_n = 1$. But, the infinite system has no solution at all!

3. Integral Equations

Each of the equations we considered above can be written in one of the forms

$$\int_a^b K(x,y)\, f(y)\, dy = g(x), \tag{1}$$

or

$$f(x) - \int_a^b K(x,y)\, f(y)\, dy = g(x), \tag{2}$$

where a, b, K, and g are all given and we want to solve for f. The function $K(x,y)$ is called the *kernel* of the equation, and we will assume that it's a reasonably "nice" function, possibly complex-valued.

One possible approach here might be to replace the integral by a sum. For example, if $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the underlying interval $[\,a,b\,]$ into n equal subintervals, and if we replace f, g, and K by suitable step functions based on these subintervals, then we might consider the "discrete" analogue of equation (2), written as a system of n equations in n unknowns

$$f_i \;-\; \frac{(b-a)}{n}\sum_{j=1}^{n} k_{i,j} f_j = g_i \qquad (i = 1,\ldots,n) \tag{3}$$

where $f_i = f(x_i)$, $g_i = g(x_i)$, and $k_{i,j} = K(x_i, x_j)$. In modern notation we might write this as

$$(I - K)f = g,$$

where $f = (f_1,\ldots,f_n)$, $g = (g_1,\ldots,g_n)$, $K = \left(\frac{(b-a)}{n}\, k_{i,j}\right)_{i,j=1,\ldots,n}$, and I is the $n \times n$ identity matrix. We would then solve this system (by techniques that were quite well known at the turn of the century) and ask whether the solutions to our "finite" problems converge to a solution of the original integral equation. We'll have more to say about this approach in a moment. And, later, we'll even give a "fancy" solution.

4. Fredholm and Hilbert

The first rigorous treatment of the general theory of integral equations was given by the Swedish astronomer and mathematician Ivar Fredholm in a series of papers in the years from 1900 to 1903. (The integral equations (1) and (2) are referred to as Fredholm equations of the first and second type, respectively.) Fredholm was intrigued by the obvious connections with systems of linear equations and, in fact, developed a theory of "determinants" for integral equations. The details would take us too far afield here, but it might be enlightening to summarize a few of Fredholm's results. To begin, we introduce a complex parameter λ and write our equation (2) as

$$f(x) - \lambda \int_a^b K(x,y)\, f(y)\, dy = g(x).$$

Fredholm defines a "determinant" $D_K(\lambda)$ associated with the kernel λK, and shows that D_K is an entire function of λ. The roots of the equation $D_K(\lambda) = 0$ are called *eigenvalues*, and the corresponding solutions to the homogeneous equation ($g(x) = 0$) are called the *eigenfunctions* of the equation. (Notice that Fredholm's eigenvalues are actually the *reciprocals* of what we would call the eigenvalues for the system; also, Fredholm gave an explicit formula for the eigenfunctions and *proved* that they were solutions to the homogeneous equation.) Further, Fredholm shows that if λ is not an eigenvalue, then the integral equation can be solved, or "inverted," by writing

$$f(x) = g(x) - \lambda \int_a^b S(x,y)\, g(y)\, dy,$$

where S is called the "resolvent kernel" (or "solving function"), and is given as the ratio of determinants—much as in Cramer's rule! What is of interest here is the fact that Fredholm used the theory of linear equations as his inspiration. In fact, he even introduced the first bit of abstract notation in this regard, by writing (2) as $A_K f(x) = g(x)$.

Now Fredholm's work was quite influential, and attracted a great deal of attention although, curiously, his techniques were largely ignored for years. This, as we'll see, is a recurring theme in the history of functional analysis! In particular, the great David Hilbert, after hearing of Fredholm's work, decided to devote his seminar to the study of integral equations. One story (from Hellinger) is that Hilbert announced that Fredholm's theory would lead to a solution to the Riemann hypothesis; he apparently felt that it would be possible to realize Riemann's zeta function as a Fredholm determinant of some appropriate integral equation. Unfortunately, no such equation was ever found.

Hilbert attacked the new theory of integral equations with a vengeance: He published a series of five papers in the years from 1904 to 1906, and a sixth in 1910; these were later collected and published under the title *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* in 1912. These papers are among the most influential papers written in our century. It would be difficult to overestimate their significance. In Hilbert's own words (but, well, translated into English!):

> . . . the systematic building of a general theory of integral equations for the whole of analysis, especially for the theory of the definite integral and the theory of the development of arbitrary functions in an infinite series, besides for the theory of linear differential equations and analytic functions, as well as for potential theory and calculus of variations, is of the greatest importance . . .

Did he leave anything out??

Hilbert took the same starting point as Fredholm: He also considered an associated "finite dimensional" systems of equations similar to (3), but with an additional complex parameter λ:

$$f_i - \lambda \sum_{j=1}^{n} k_{i,j}\, f_j = g_i \qquad (i = 1, \ldots, n) \tag{4}$$

Rather than writing down a clever solution and verifying it, as Fredholm had done, Hilbert made rigorous the passage to the limit. In so doing, he was able to prove analogues of many familiar results from the theory of linear equations in this new setting of integral equations.

In a nutshell, the importance of Hilbert's contribution is that he completely abandoned the integral equations (!) in favor of the assumption that the theory should be nothing more than a special case of what was already known about systems of linear equations. As some authors would say, Hilbert began the "algebraization" of analysis!

To begin, Hilbert converts the system (4) into a system involving bilinear forms: By introducing the notation

$$(x, y) = \sum_{i=1}^{n} x_i\, y_i$$

for the inner product of two vectors x and y, the system of equations

$$f - \lambda K f = g \tag{4$'$}$$

is written as

$$(u, f) - \lambda\, (u, Kf) = (u, g) \tag{5}$$

where now the vector f will be considered a solution if equation (5) is satisfied for every vector u.

Hilbert solved (4$'$) in much the same way that Fredholm had, but with a twist: In the special case of a *symmetric* kernel; that is, the case where $K(x, y)$ satisfies $K(x, y) = K(y, x)$, Hilbert was able to develop a more complete theory. In particular, he established an analogy between the bilinear form

$$(Kx, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{i,j}\, x_i\, y_i$$

and the integral form

$$\int_a^b \int_a^b K(s,t)\, x(s)\, y(t)\, ds\, dt.$$

In this case, the eigenvalues of the integral equation are shown to be a sequence of real numbers (λ_n) and, just as you might guess, the eigenfunctions corresponding to distinct eigenvalues are *orthogonal*. That is, if φ_n is the eigenfunction corresponding to the eigenvalue λ_n, then

$$\varphi_n(x) = \lambda_n \int_a^b K(x,y)\, \varphi_n(y)\, dy,$$

and, for $n \neq m$,

$$\int_a^b \varphi_n(x)\, \varphi_m(x)\, dx = 0.$$

What's more, the φ_n's can be *normalized*, meaning that we may also assume that

$$\int_a^b \big(\varphi_n(x)\big)^2\, dx = 1.$$

With all this notation at our disposal, we can state a couple of Hilbert's main results. The first of these should be viewed as an extension of the principal axis theorem to integral equations (which is precisely what Hilbert had in mind).

Theorem. *Let $K(s,t)$ be a continuous, symmetric kernel, and let φ_n be the normalized eigenfunction corresponding to the eigenvalue λ_n of the integral equation*

$$f(s) - \lambda \int_a^b K(s,t)\, f(t)\, dy = g(s).$$

Then, for any continuous functions $x(s)$ and $y(s)$, we have

$$\int_a^b\!\!\int_a^b K(s,t)\, x(s)\, y(t)\, ds\, dt = \sum_{n=1}^{m} \frac{1}{\lambda_n} \left(\int_a^b \varphi_n(s)\, x(s)\, ds\right)\!\left(\int_a^b \varphi_n(s)\, y(s)\, ds\right)$$

where m is finite or infinite, according to the number of eigenvalues and where, in the latter case, the series converges absolutely and uniformly for any x and y satisfying

$$\int_a^b x(s)^2 ds < \infty \quad \text{and} \quad \int_a^b y(s)^2 ds < \infty.$$

In modern notation this result reads

$$(Kx, y) = \sum_n \frac{1}{\lambda_n} (\varphi_n, x)\, (\varphi_n, y).$$

where

$$(Kx)(s) = \int_a^b K(s, t)\, x(t)\, dt,$$

and

$$(x, y) = \int_a^b x(s)\, y(s)\, ds.$$

The Hilbert-Schmidt Theorem. *If $f(x)$ satisfies*

$$f(x) = \int_a^b K(x, y)\, g(y)\, dy$$

for some continuous $g(x)$, then $f = \sum_n c_n \varphi_n$, where (φ_n) are the orthonormal eigenfunctions for K and where $c_n = \int_a^b \varphi_n(x)\, f(x)\, dx$.

Here is Hilbert's connection with Fourier series. A series written in terms of orthogonal functions is sometimes called a "generalized" Fourier series—Hilbert even referred to the numbers c_n as the Fourier coefficients of f relative to the sequence (φ_n). Note that in our notation $c_n = (\varphi_n, f)$.

We will return to a discussion of Hilbert's work, and its followers, after we bring Maurice Fréchet into the story. For this we'll need to know just a little about the Calculus of variations.

5. The Calculus of Variations

Just imagine how complicated the calculations of Fredholm and Hilbert must have looked—especially since they insisted on writing out all those integrals! It took a giant leap of intuition, I would think, to even consider the equations without the extra notation that was habitually included. After all, it was not at all commonplace to even call functions by "name"; the common practice was, rather, to think of functions as "formulas." That is, to consider functions in terms of their *values* and not as generic "mappings" of one set into another. In other words, it was considered important to display the *arguments* of a function and so, in the case of a "function of a function," this could get quite cumbersome.

As a very simple example in this regard, consider the maximum value of a continuous function defined on a closed interval

$$M(f) = \max_{a \leq x \leq b} |f(x)|.$$

Clearly, M is a well-defined "function of (certain) functions," but what are the *arguments* of M? Is it possible to give a *formula* for $M(f)$?? (The answer, as it happens, is yes!)

[Notice, please, that $M(f)$ isn't really well-defined for <u>all</u> functions. It was already commonplace at the turn of the century to consider only certain "classes" or "sets" of functions, but without regard to any common structural properties. And certainly such "classes" were not viewed as "spaces" of functions—this would come later.]

To further complicate things, suppose that among all "curves" $y = f(x)$ satisfying $f(a) = 0$ and $f(b) = 1$, we seek the one whose graph generates a surface of revolution with smallest possible surface area

$$A(f) = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx.$$

That is, we want to minimize $A(f)$. Now, what are the "coordinates" of A?? I'll need to know, because in order to find a minimum value I'll want to find a "zero derivative" for A!! (Really!!)

Finding maximum and minimum values for such "functions of functions" is the domain of the Calculus of variations. Other examples are the area under the graph of a function, arc length, and so on. In short, we're interested in what Volterra called "functions of lines." Functions whose arguments are other functions, or "curves." Hadamard suggested a different name: He referred to these "special" functions as *fonctionelles*, which later became *functionals*, and he referred to the Calculus of variations as "the analysis of functionals," or "functional analysis." (Paul Lévy was apparently the first to use the phrase "functional analysis," in 1922.) The word functional has come to mean something slightly different, but it's interesting to see the connection with its origins. Volterra's work proved to be somewhat unsatisfactory, so we won't say much more about him, but Hadamard is a key player in our story: He was Fréchet's adviser and Fréchet's work makes up the next chapter of our tale.

Two Italian mathematicians are important in this regard. G. Ascoli (1883) and C. Arzelà (1889) both worked on the problem of extending Cantor's set theory to include the new notion of "sets of functions." For example, both Ascoli and Arzelà considered the problem of giving necessary and sufficient conditions for the existence of a uniformly convergent subsequence of a given sequence of functions; of particular interest to them was the problem of interchanging limits and (Riemann) integrals:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$$

It was in this context that the so-called Ascoli-Arzelà theorem was born (actually, the version we are familiar with is due to Arzelà, but the notion of equicontinuity is due to Ascoli).

6. Fréchet and Abstract Spaces

Unarguably, the most vocal and influential proponent of abstraction at the turn of the century had to be Fréchet. And it's precisely the problem of designing a "coordinate free" brand of the new "functional analysis" that Fréchet was most interested in.

In referring to the methods of Volterra, Hilbert, and others (who had used brute force methods of passing from the finite to the infinite) Fréchet had some rather strong views; here is what he had this to say about such methods in 1928:

> We believe that this method has played an important role in seconding intuition, but that its time has ended. It is a useless artifice to substitute for a function an infinite sequence of numbers which, moreover, may be chosen in a variety of fashions. This is quite evident, for example, in the theory of integral equations where the solutions of Fredholm and Schmidt are much simpler and more elegant than those of Hilbert, which is not to take away from the latter the essential merit of having obtained a great number of new results.

Fréchet advocated a new “General Analysis,” or “Functional Calculus,” which was based on two very general principles:

— Basic notions from set theory, and

— a notion of *limit* (which was assumed to be available in the particular class of “spaces” he considered—more on this in a moment).

In modern terms, Fréchet’s “General Analysis” was an early example of what we would now call point set topology. In particular, it was Fréchet who formalized the notion of a metric space—and was among the first to use the word “space,” for that matter, as a word meaning an abstract (or “indifferent”) set that carries with it some additional structure. The main results in his 1906 thesis were generalizations of the work of

▷ Cantor (by generalizing the notions of the interior of a set, the derived set, compactness, perfect set, and many more),

▷ Baire (by generalizing the notion of semi-continuous functions, for example), and

▷ Arzelà (by extending the notion of compactness to sets of functions).

Please note that 1906 is precisely the same year that Hilbert’s influential papers appeared. What this means is that Frèchet’s work is the only major source for the later development of function spaces (and other abstract spaces) that did not stem from Hilbert’s work.

Several important examples are associated with Fréchet's name, although certain details of these examples were well known at the time. The first is the space $C[\,a,b\,]$, consisting of all continuous, real-valued functions on the interval $[\,a,b\,]$ together with the distance function

$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|.$$

It was well-known that convergence in this metric is the same as uniform convergence, for example, but it was Fréchet's investigations into the finer structure of this collection that led to his generalization of Arzelá's results. Another example is "the space of countably infinite dimension," which Fréchet writes as E_ω, which is defined to be the space of all sequences (x_n) of real numbers, together with the metric

$$\rho\big((x_n),(y_n)\big) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

And, lastly, a space we might write as $C(\mathbf{R})$, the collection of all continuous functions real-valued functions on $\mathbf{R}$ under the metric

$$D(f,g) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{d_n(f,g)}{1 + d_n(f,g)},$$

where d_n refers to the metric on $C[\,-n,n\,]$. Each of these last two metrics are sometimes called "Fréchet's metric."

Fréchet's approach was so new, so revolutionary, that he felt the need to justify it at every opportunity—even as late as 1950!! But he did not generalize for the sake of generalization—indeed, he abhorred such practices—instead, his abstractions were firmly based on the premise that abstraction would lend a fresh point of view to old results.

It's interesting to note, too, that much of Fréchet's terminology is still with us—you would find much of his work quite easy to follow. But, sadly, Fréchet will most likely never be seen as a major influence: To modern eyes his results seem routine and deadly dull. For this reason his work is not often seen for what is is: A major influence on modern mathematics. After all, hundreds of mathematicians were trained using his 1928 book *Les espaces abstraits*, along with *Mengenlehre*, 1914, from Hausdorff, whose work was strongly influenced by Fréchet's.

(At about the same time as Fréchet's thesis, the American mathematician E. H. Moore was espousing a different flavor of "general analysis." However, rather than helping Fréchet's cause, Moore's work may have slowed the acceptance of Fréchet's brand of analysis. Moore's work was not well received: It was overly complicated and difficult to read, and his results were not all they might be. In short, his papers were more trouble than they were worth—at least that was the view of Hellinger and Toeplitz in an important article they had written on integral equations for the *Encyklopädie der Mathematischen Wissenschaften.*)

7. Hilbert's Successors

Hilbert went on to study infinite bilinear forms of the type

$$K(x,y) = \sum_{i,j=1}^{\infty} k_{i,j}\, x_i\, y_j,$$

where $x = (x_i)$ and $y = (y_i)$ are sequences satisfying

$$\sum_{i=1}^{\infty} x_i^2 < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} y_i^2 < \infty.$$

We won't pursue these results further, but it should be clear enough by now that Hilbert had established a wide variety of "algebraic" tools for use in the setting of integral equations. In particular, he developed more than enough machinery to justify the careful study of "spaces" of square-summable sequences together with the inner product

$$(x,y) = \sum_{i=1}^{\infty} x_i\, y_i,$$

and their "continuous" analogue: The "space" of square-integrable functions together with the inner product

$$\int_a^b x(s)\, y(s)\, ds.$$

(Later, these inner products were changed slightly by using $\overline{y_i}$ and $\overline{y(s)}$. For our purposes, though, there's no harm in assuming that we're speaking of real-valued functions.) It's interesting to note, however, that there is no record that Hilbert ever considered any such "spaces." Indeed, this particular development would be left to Schmidt, Fréchet, Riesz, and Fischer. In fact, the first use of the words "Hilbert space" (or, rather, *espace de Hilbert*) is due to Riesz in 1913, from his book on systems of equations in infinitely many unknowns.

In 1907, Erhard Schmidt (this is the Schmidt of the "Gram-Schmidt process") introduced what he called "function spaces." In modern terminology, Schmidt developed the general theory of the space we would call ℓ_2, the collection of all sequences (z_j) of complex numbers satisfying

$$\sum_{j=1}^{\infty} |z_j|^2 < \infty,$$

and endowed with the inner product

$$(z, w) = \sum_{j=1}^{\infty} z_j\, \overline{w_j}.$$

Schmidt further introduces (possibly for the first time) the double bar notation $\|z\|$ to denote the *norm* of z, defined by

$$\|z\|^2 = (z, z) = \sum_{j=1}^{\infty} z_j\, \overline{z_j} = \sum_{j=1}^{\infty} |z_j|^2 < \infty.$$

Schmidt goes on to consider various types of convergence in his new function space and, in particular, considers the notion of a closed subspace. Here we find one of Schmidt's most important contributions: *The Projection Theorem.*

Also in 1907, both Schmidt and Maurice Fréchet remarked that the space $L_2[\,a,b\,]$, consisting of all those (Lebesgue measurable) functions f on $[\,a,b\,]$ for which

$$\int_a^b \big(f(x)\big)^2 dx < \infty,$$

supported a geometry completely analogous to Schmidt's space of square summable sequences.

Meanwhile, in a series of papers from 1907, the great Hungarian mathematician Friedrich Riesz investigated the collection of (Lebesgue) square-integrable functions, a space Riesz would later refer to as L_2. Riesz was motivated in this by Hilbert's work, and also by the recent introduction of the Lebesgue integral (1902), an important paper of Pierre Fatou which applied the new integral (1906), and Fréchet's work on abstract spaces (1906, and the 1907 result cited above). An important contribution is the following:

Theorem. *Let (φ_n) be an orthonormal sequence of square integrable functions defined on an interval $[\,a,b\,]$, and let (a_n) be a sequence of real numbers. Then, the condition*

$$\sum_{n=1}^{\infty} a_n^2 < \infty$$

is both necessary and sufficient for the existence of a square-integrable function f satisfying

$$\int_a^b f(x)\,\varphi_n(x)\,dx = a_n \quad \textit{for all} \quad n.$$

What Riesz's result tells us is that there is a one-to-one correspondence between Schmidt's space ℓ_2 and the space L_2 (by means of an intermediary orthonormal sequence (φ_n)).

At nearly the same time, Ernst Fischer considered the notion of *convergence in mean* for square-summable functions. A sequence (f_n) is said to converge in mean to a function f if

$$\lim_{n\to\infty} \int_a^b \big(f_n(x) - f(x)\big)^2 dx = 0.$$

Fischer's most important result is, in modern language, the fact that L_2 is *complete* with respect to convergence in mean. From this, Fischer deduced Riesz's result, above, and the combined result is usually referred to as the Riesz-Fischer theorem. Today this result is viewed as a remarkable discovery but, at the time, it was considered a mere technical observation in a very specialized area.

8. Hahn, Helly, Banach, and Normed Vector Spaces

Let's turn the clock ahead to 1922 and give an all too brief discussion of the contributions of Eduard Helly, Hans Hahn, and the great Polish mathematician Stefan Banach. Especially Banach.

While Helly and Hahn are important players in the story of functional analysis, making several important contributions to its early development, it was Banach who gave the first *complete* treatment of abstract normed vector spaces. And it's this very word *complete* that must be emphasized!!

Banach's thesis, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrals*, appeared in 1922 in the third volume of Sierpiński's *Fundamenta Mathematicae*. In it, Banach discusses several diverse and important applications of the new theory of "functionals" (which, if you'll recall, meant functions whose domain or range is a set of functions). In Banach's own words (modulo translation, of course):

> However, in order not to have to demonstrate [certain theorems] separately for each set [of examples] . . . I have chosen a different way: I consider in a general fashion the set of elements on which I postulate certain properties, I deduce theorems on it, and then show for each set of particular functionals that the . . . postulates are true for it.

This is hardly a new approach to us—but just consider: Banach felt the need to justify this highly abstract approach as late as 1922!

Of course, most of us are familiar with the notion of a Banach space, which was introduced (in its full glory, that is) in Banach's thesis. But, just to be on the safe side: Banach introduces the axioms for a vector space X (these were known at the time, but were apparently not considered well-known), and assumes, further, that the space X carries a *norm.* That is, a function $\|\cdot\|$ defined on X satisfying

1. $\|x\| \geq 0$ for every $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \, \|x\|$ for every $x \in X$ and $\alpha \in \mathbf{R}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

and a fourth assumption, that of completeness

4. If $\lim\limits_{m,n\to\infty} \|x_n - x_m\| = 0$, then there exists an $x \in X$ such that $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. That is, if (x_n) is a Cauchy sequence in X, then (x_n) converges in norm to some element of X.

As one of his first results, Banach gives a new characterization of completeness for normed linear spaces (this should be viewed as a generalization of the so-called "Weierstrass M-test"):

Theorem. *The normed vector space X is complete if and only if the condition $\sum_n \|x_n\| < \infty$ always implies that the series $\sum_n x_n$ converges in the norm of X. That is, X is complete if and only if every absolutely summable series in X is summable.*

From these various axioms, Banach goes on to deduce a full battery of results concerning the topology of his spaces and the continuous "operations" on them. In particular, of course, Banach is interested in linear functions F defined on X which are continuous in the sense that

$$\lim_{n\to\infty} F(x_n) = F(x) \quad \text{whenever} \quad \lim_{n\to\infty} x_n = x.$$

Banach proves, for example, that a linear function F on X is continuous precisely when it is *bounded*; that is, when there exists a constant M, depending only on F, such that $\|F(x)\| \le M\|x\|$ holds for all $x \in X$.

It is also in this paper that Banach proves his *contraction mapping theorem.* Rather than state this familiar theorem, let's consider one of Banach's own applications of the result: We'll give another solution to Fredholm's integral equation of the second kind (2) which is both simple and enlightening.

An Example

Let's agree to write the integral equation (2) as

$$f = g + Kf, \tag{2''}$$

where $(Kf)(x) = \int_a^b K(x,y) f(y)\, dy$. That is, we'll think of this integral as a linear operator and bask in the glory of "letter juggling"! The method we'll use is sometimes called *the method of successive approximations*—it dates back to Liouville, at least, but was probably known even to Cauchy. In this method, we make a first "guess" at a solution; say, $f_0 = 0$, and consider

$$f_1 = g + Kf_0 = g$$

as a good second "guess" at a solution. Continuing, we consider

$$f_2 = g + Kf_1 = g + Kg, \quad f_3 = g + Kf_2 = g + Kg + K^2 g, \quad \ldots$$

and so on, where

$$\begin{aligned}(K^2 g)(x) = K(Kg)(x) &= \int_a^b K(x,y) \int_a^b K(y,z)\, g(z)\, dz\, dy \\ &= \int_a^b \int_a^b K(x,y)\, K(y,z)\, g(z)\, dz\, dy\end{aligned}$$

is the iterated operator. Thus, we're led to consider the "Neumann series"

$$g + Kg + K^2 g + \cdots + K^n g + \cdots.$$

It's easy to see that if this series converges *uniformly* to some function f, then f is a solution to our integral equation. Indeed, in this case term-by-term integration of the series is allowed, and so we have

$$g + Kf = g + K(g + Kg + K^2g + \cdots) = g + Kg + K^2g + K^3g + \cdots = f.$$

The name "Neumann series" is after Carl Neumann, 1877, who gave the first rigorous proof that this series converges (under suitable hypotheses on K and g).

Now this example is very meaningful for our purposes: One obvious point here is the fact that modern notation frees the problem from the distracting involvement of the underlying variable "x." This alone is quite a modern notion! But also notice that once we've reduced the problem to "letter juggling," it's easier to tell what ingredients are needed to make it work. For example, we certainly used the fact that integration against $K(x,y)$ is linear and "continuous" (we interchanged a uniform limit and integration). And simple assumptions on $K(x,y)$ will insure the uniform convergence of the series; for instance, suppose we write

$$M = \max_{a \le x \le b} \int_a^b |K(x,y)|\,dy.$$

Then,

$$|(Kg)(x)| = \left| \int_a^b K(x,y)g(y)\,dy \right| \le M \cdot \max_{a \le y \le b} |g(y)|.$$

That is, in terms of the norm on $C[\,a,b\,]$,

$$\|Kg\| \le M\,\|g\|.$$

From this it follows easily that

$$\|K^2g\| = \|K(Kg)\| \le M^2\|g\|,$$

and so on. In general, $\|K^n g\| \le M^n \|g\|$. Thus, if we should be so fortunate as to have $M < 1$, then we would also have

$$\sum_{n=1}^{\infty} \|K^n g\| \le \|g\| \sum_{n=1}^{\infty} M^n < \infty.$$

Consequently, from Banach's version of the M-test, our "Neumann series" will converge uniformly to a solution of $(2'')$.

Better still, Banach takes this approach to the method of successive approximations a step further. Notice that if we let the letters do their magic, then it's even possible to see why the method works:

$$f - Kf = g \Longrightarrow (I - K)f = g \Longrightarrow f = (I - K)^{-1} g,$$

and

$$(I - K)^{-1} = \frac{1}{I - K} = I + K + K^2 + K^3 + \cdots \quad !$$

(What else could it be??)

In the parlance of operators, Banach's completeness theorem shows that if K is a continuous linear operator between Banach spaces, and if $\|K\| < 1$ (where this is the so-called operator norm), then $I - K$ has a continuous inverse which may be written as a power series in K (and the series is even absolutely summable in operator norm). [From this observation, it follows that the set of invertible operators is an *open* subset in the space of operators.]

What's more, if we rewrite this calculation using the extra parameter λ, we even get Banach's version of the spectral radius theorem:

Theorem. *Let K be a continuous linear operator on a Banach space X, and suppose that $\|Kx\| \leq M\,\|x\|$ for all x in X. Then, for any λ with $|\lambda M| < 1$, and any y in X, the equation $x + \lambda Kx = y$ always has a unique solution. Moreover, the solution may be written as an absolutely summable series*

$$x = y + \sum_{n=1}^{\infty} (-1)^n \lambda^n K^n y.$$

This formula may not ring any bells for us, but it is very similar to one of Fredholm's formulas. Banach's contemporaries would recognize this as a statement about integral equations.

If you recall that Banach's parameter λ is the reciprocal of what we would use, then we have a version of the spectral radius theorem: That is, the operator $(\lambda I - K)$ is invertible whenever $|\lambda| > \|K\|$. Thus, the (modern) eigenvalues of K are contained in the ball of radius $\|K\|$ in the complex plane.

9. References

I have included here a few original works and several secondary sources. I would especially recommend the two papers by Bernkopf (which are summarized in Kline's book) and the article by Taylor.

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