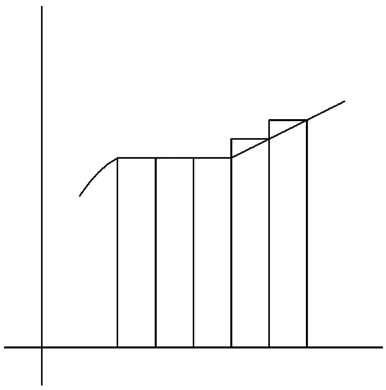
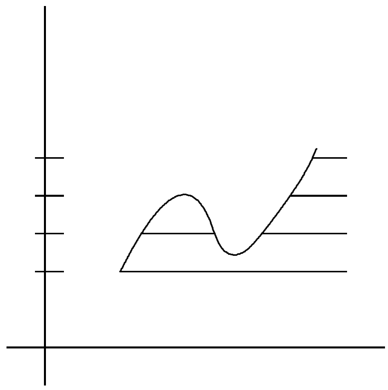
Lebesgue Integration on $\mathbb{R}^n$

The treatment here is based loosely on that of Jones, *Lebesgue Integration on Euclidean Space*. We give an overview from the perspective of a user of the theory.

Riemann integration is based on subdividing the *domain* of f. This leads to the requirement of some "smoothness" of f for the Riemann integal to be defined: for x, y close, $f(x)$ and $f(y)$ need to have something to do with each other. Lebesgue integration is based on subdividing the *range space* of f: it is built on inverse images.

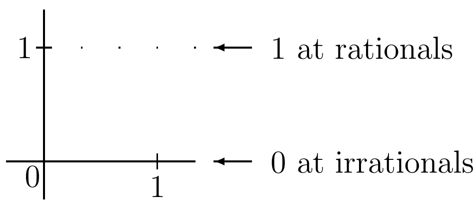




Typical Example. For a set $E \subset \mathbb{R}^n$, define the characteristic function of the set E to be

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases} .$$

Consider $\int_0^1 \chi_{\mathbb{Q}}(x)dx$, where $\mathbb{Q} \subset \mathbb{R}$ is the set of rational numbers:



Riemann: The upper Riemann integral is the inf of the "upper sums": $\overline{\int_0^1} \chi_{\mathbb{Q}}(x)dx = 1$.

The lower Riemann integral is the sup of the "lower sums": $\underline{\int_0^1} \chi_{\mathbb{Q}}(x)dx = 0$.

Since $\overline{\int_0^1} \chi_{\mathbb{Q}}(x)dx \neq \underline{\int_0^1} \chi_{\mathbb{Q}}(x)dx$, $\chi_{\mathbb{Q}}$ is *not* Riemann integrable.

Lebesgue: Let $\lambda(E)$ denote the Lebesgue measure ("size") of E (to be defined). Then

$$\begin{aligned}\int_0^1 \chi_{\mathbb{Q}}(x)dx &= 1 \cdot \lambda(\mathbb{Q} \cap [0,1]) + 0 \cdot \lambda(\mathbb{Q}^c \cap [0,1]) \\ &= 1 \cdot 0 + 0 \cdot 1 = 0.\end{aligned}$$

First, we must develop the theory of Lebesgue measure to measure the "size" of sets.

Advantages of Lebesgue theory over Riemann theory:

1. Can integrate more functions (on finite intervals).
2. Good convergence theorems: $\lim_{n\to\infty} \int f_n(x)dx = \int \lim_{n\to\infty} f_n(x)dx$ under mild assumptions.
3. Completeness of L^p spaces.

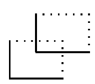
Our first task is to construct Lebesgue measure on $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$, we want to define $\lambda(A)$, the Lebesgue measure of A, with $0 \leq \lambda(A) \leq \infty$. This should be a version of n-dimensional volume for general sets. However, it turns out that one can't define $\lambda(A)$ for *all* subsets $A \subset \mathbb{R}^n$ and maintain all the desired properties. We will define $\lambda(A)$ for "[Lebesgue] measurable" subsets of $\mathbb{R}^n$ (very many subsets).

We define $\lambda(A)$ for increasingly complicated sets $A \subset \mathbb{R}^n$. See Jones for proofs of the unproved assertions made below.

Step 0. Define $\lambda(\emptyset) = 0$.

Step 1. We call a set $I \subset \mathbb{R}^n$ a special rectangle if $I = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$, where $-\infty < a_j < b_j < \infty$. (Note: Jones leaves the right ends closed). Define $\lambda(I) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Step 2. We call a set $P \subset \mathbb{R}^n$ a special polygon if P is a finite union of special rectangles.



Fact: Every special polygon is a disjoint union of finitely many special rectangles.

For $P = \bigcup_{k=1}^{N} I_k$, where the I_k's are disjoint (i.e., for $j \neq k$, $I_j \cap I_k = \emptyset$), define $\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$. Note that a special polygon may be written as a disjoint union of special rectangles in different ways.

Fact: $\lambda(P)$ is independent of the way that P is written as a disjoint union of special rectangles.

Step 3. Let $G \subset \mathbb{R}^n$ be a nonempty open set. Define

$$\lambda(G) = \sup\{\lambda(P) : P \text{ is a special polygon, } P \subset G\}.$$

(Approximation by special polygons from the inside.)

Remark: Every nonempty open set in $\mathbb{R}^n$ can be written as a *countable* disjoint union of special rectangles.

Step 4. Let $K \subset \mathbb{R}^n$ be compact. Define

$$\lambda(K) = \inf\{\lambda(G) : G \text{ open, } K \subset G\}.$$

(Approximation by open sets from the outside.)

Fact: If $K = \overline{P}$ for a special polygon P, then $\lambda(K) = \lambda(P)$.

Now for $A \subset \mathbb{R}^n$, A arbitrary, define

$$\begin{aligned} \lambda^*(A) &= \inf\{\lambda(G) : G \text{ open, } A \subset G\} \qquad (\textit{outer} \text{ measure of } A) \\ \lambda_*(A) &= \sup\{\lambda(K) : K \text{ compact, } K \subset A\} \qquad (\textit{inner} \text{ measure of } A) \end{aligned}$$

Facts: If A is open or compact, then $\lambda_*(A) = \lambda(A) = \lambda^*(A)$. Hence for any A, $\lambda_*(A) \leq \lambda^*(A)$.

Step 5. A bounded set $A \subset \mathbb{R}^n$ is said to be [Lebesgue] measurable if $\lambda_*(A) = \lambda^*(A)$. In this case we define $\lambda(A) = \lambda_*(A) = \lambda^*(A)$.

Step 6. An arbitrary set $A \subset \mathbb{R}^n$ is said to be [Lebesgue] measurable if for each $R > 0$, $A \cap B(0, R)$ is measurable, where $B(0, R)$ is the open ball of radius R with center at the origin. If A is measurable, define $\lambda(A) = \sup_{R>0} \lambda(A \cap B(0, R))$.

Let $\mathcal{L}$ denote the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$.

Fact. $\mathcal{L}$ is a σ-algebra of subsets of $\mathbb{R}^n$. That is, $\mathcal{L}$ has the properties:

(i) $\emptyset$, $\mathbb{R}^n \in \mathcal{L}$.

(ii) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$.

(iii) If $A_1, A_2, \ldots \in \mathcal{L}$ is a *countable* collection of subsets of $\mathbb{R}^n$ in $\mathcal{L}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{L}$.

Fact. If $\mathcal{S}$ is any collection of subsets of a set X, then there is a smallest σ-algebra $\mathcal{A}$ of subsets of X containing $\mathcal{S}$ (i.e., with $\mathcal{S} \subset \mathcal{A}$), namely, the intersection of all σ-algebras of subsets of X containing $\mathcal{S}$. This smallest σ-algebra $\mathcal{A}$ is called the σ-algebra *generated* by $\mathcal{S}$.

Definition. The smallest σ-algebra of subsets of $\mathbb{R}^n$ containing the open sets is called the collection $\mathcal{B}$ of *Borel sets.* Closed sets are Borel sets.

Fact. Every open set is [Lebesgue] measurable. Thus $\mathcal{B} \subset \mathcal{L}$.

Fact. If $A \in \mathcal{L}$, then $\lambda_*(A) = \lambda(A) = \lambda^*(A)$.

Caution: However, $\lambda_*(A) = \lambda^*(A) = \infty$ does *not* imply $A \in \mathcal{L}$.

Properties of Lebesgue measure

λ is a *measure.* This means:

1. $\lambda(\emptyset) = 0$.
2. $(\forall\, A \in \mathcal{L})\ \lambda(A) \geq 0$.
3. If $A_1, A_2, \ldots \in \mathcal{L}$ are *disjoint* then $\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k)$. (*countable additivity*)

Consequences:

(i) If $A, B \in \mathcal{L}$ and $A \subset B$, then $\lambda(A) \leq \lambda(B)$.

(ii) If $A_1, A_2, \cdots \in \mathcal{L}$, then $\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$. (*countable subadditivity*)

Remark: Both (i) and (ii) are true of outer measure λ^* on *all* subsets of $\mathbb{R}^n$.

Sets of Measure Zero

Fact. If $\lambda^*(A) = 0$, then $0 \leq \lambda_*(A) \leq \lambda^*(A) = 0$, so $0 = \lambda_*(A) = \lambda^*(A)$, so $A \in \mathcal{L}$. Thus every subset of a set of measure zero is also measurable (we say λ is a *complete measure*).

Characterization of Lebesgue measurable sets

Definition. A set is called a G_δ if it is the intersection of a countable collection of open sets. A set is called an F_σ if it is the union of a countable collection of closed sets. G_δ sets and F_σ sets are Borel sets.

Fact. A set $A \subset \mathbb{R}^n$ is Lebesgue measurable iff $\exists$ a G_δ set G and an F_σ set F for which $F \subset A \subset G$ and $\lambda(G \backslash F) = 0$. (Note: $G \backslash F = G \cap F^c$ is a Borel set.)

Examples.

(0) If $A = \{a\}$ is a single point, then $A \in \mathcal{L}$ and $\lambda(A) = 0$.

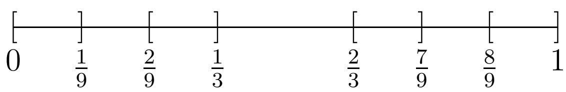
(1) If $A = \{a_1, a_2, \ldots\}$ is countable, then A is measurable, and $\lambda(A) \leq \sum_{j=1}^{\infty} \lambda(\{a_j\}) = 0$, so $\lambda(A) = 0$. For example, $\lambda(\mathbb{Q}) = 0$.

(2) $\lambda(\mathbb{R}^n) = \infty$.

(3) **Open sets in** $\mathbb{R}$. Every nonempty open set $G \subset \mathbb{R}$ is a (finite or) countable *disjoint* union of open intervals (a_j, b_j) ($1 \leq j \leq J$ or $1 \leq j < \infty$), and $\lambda(G) = \sum_j \lambda(a_j, b_j) = \sum_j (b_j - a_j)$.

(4) **The Cantor Set** is a closed subset of $[0, 1]$. Let

$$\begin{aligned} G_1 &= \left(\frac{1}{3}, \frac{2}{3}\right), \qquad \lambda(G_1) = \frac{1}{3} \\ G_2 &= \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right), \qquad \lambda(G_2) = \frac{2}{9} \\ G_3 &= \left(\frac{1}{27}, \frac{2}{27}\right) \cup \cdots \cup \left(\frac{25}{27}, \frac{26}{27}\right), \qquad \lambda(G_3) = \frac{4}{27} \\ \text{etc.} & \quad \left(\text{note } \lambda(G_k) = \frac{2^{k-1}}{3^k}\right) \end{aligned}$$



(middle thirds of remaining subintervals)

Let $G = \bigcup_{k=1}^{\infty} G_k$, so G is an open subset of $(0, 1)$. Then

$$\lambda(G) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{1}{3}\frac{1}{(1 - \frac{2}{3})} = 1.$$

Define the Cantor set $C = [0, 1] \backslash G$. Since $\lambda(C) + \lambda(G) = \lambda([0, 1]) = 1$, we have $\lambda(C) = 0$.

Fact. For $x \in [0,1]$, $x \in C$ iff x has a base 3 expansion with only 0's and 2's, i.e., $x = \sum_{j=1}^{\infty} d_j 3^{-j}$ with each $d_j \in \{0,2\}$.

For example:
$$\begin{aligned} 0 &= (0.000\cdots)_3 \\ \tfrac{1}{3} &= (0.100\cdots)_3 = (0.0222\cdots)_3 \\ \tfrac{2}{3} &= (0.200\cdots)_3 \\ 1 &= (0.222\cdots)_3 \end{aligned}$$

$\frac{3}{4} = (0.202020\cdots)_3$ is in C, but it is not an endpoint of any interval in any G_k. Despite the fact that $\lambda(C) = 0$, C is not countable. In fact, C can be put in $1{-}1$ correspondence with $[0,1]$ (and thus also with $\mathbb{R}$).

Invariance of Lebesgue measure

(1) **Translation**. For a fixed $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, define $x + A = \{x + y : y \in A\}$.

Fact. If $x \in \mathbb{R}^n$ and $A \in \mathcal{L}$, then $x + A \in \mathcal{L}$, and $\lambda(x + A) = \lambda(A)$.

(2) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $A \in \mathcal{L}$, then $T(A) \in \mathcal{L}$, and $\lambda(T(A)) = |\det T| \cdot \lambda(A)$.

Measurable Functions

We consider functions f on $\mathbb{R}^n$ with values in the extended real numbers $[-\infty, \infty]$. We extend the usual arithmetic operations from $\mathbb{R}$ to $[-\infty, \infty]$ by defining $x \pm \infty = \pm\infty$ for $x \in \mathbb{R}$; $a \cdot (\pm\infty) = \pm\infty$ for $a > 0$; $a \cdot (\pm\infty) = \mp\infty$ for $a < 0$; and $0 \cdot (\pm\infty) = 0$. The expressions $\infty + (-\infty)$ and $(-\infty) + \infty$ are usually undefined, although we will need to make some convention concerning these shortly. A function $f : \mathbb{R}^n \to [-\infty, \infty]$ is called *Lebesgue measurable* if for every $t \in \mathbb{R}$, $f^{-1}([-\infty, t]) \in \mathcal{L}$ (in $\mathbb{R}^n$).

Recall: Inverse images commute with unions, intersections, and complements:

$$f^{-1}[B^c] = f^{-1}[B]^c, \qquad f^{-1}\left[\bigcup_\alpha A_\alpha\right] = \bigcup_\alpha f^{-1}[A_\alpha], \qquad f^{-1}\left[\bigcap_\alpha A_\alpha\right] = \bigcap_\alpha f[A_\alpha].$$

Fact. For any function $f : \mathbb{R}^n \to [-\infty, \infty]$, the collection of sets $B \subset [-\infty, \infty]$ for which $f^{-1}[B] \in \mathcal{L}$ is itself a σ-algebra of subsets of $[-\infty, \infty]$.

Note. The smallest σ-algebra of subsets of $[-\infty, \infty]$ containing all sets of the form $[-\infty, t]$ for $t \in \mathbb{R}$ contains also $\{-\infty\}$, $\{\infty\}$, and all sets of the form $[-\infty, t)$, $[t, \infty]$, $(t, \infty]$, (a, b), etc. It is the collection of all sets of the form B, $B \cup \{\infty\}$, $B \cup \{-\infty\}$, or $B \cup \{-\infty, \infty\}$ for Borel subsets B of $\mathbb{R}$.

Comments. If f and $g : \mathbb{R}^n \to [-\infty, \infty]$ are measurable, then $f + g$, $f \cdot g$, and $|f|$ are measurable. (Here we need to make a convention concerning $\infty + (-\infty)$ and $(-\infty) + \infty$. This statement concerning measurability is true so long as we define both of these expressions to be the same, arbitrary but fixed, number in $[-\infty, \infty]$. For example, we may define $\infty + (-\infty) = (-\infty) + \infty = 0$.) Moreover, if $\{f_k\}$ is a sequence of measurable functions $f_k : \mathbb{R}^n \to [-\infty, \infty]$, then so are $\sup_k f_k(x)$, $\inf_k f_k(x)$, $\underbrace{\limsup_k f_k(x)}_{=\inf_{k\geq 1}\sup_{j\geq k} f_j(x)}$, $\liminf_k f_k(x)$. Thus if $\lim_{k\to\infty} f_k(x)$ exists $\forall\, x$, it is also measurable.

Definition. If $A \subset \mathbb{R}^n$, $A \in \mathcal{L}$, and $f : A \to [-\infty, \infty]$, we say that f is measurable (on A) if, when we extend f to be 0 on A^c, f is measurable on $\mathbb{R}^n$. Equivalently, we require that $f\chi_A$ is measurable for any extension of f.

Definition. If $f : \mathbb{R}^n \to \mathbb{C}$ (*not* including ∞), we say f is Lebesgue measurable if $\mathcal{R}ef$ and $\mathcal{I}mf$ are both measurable.

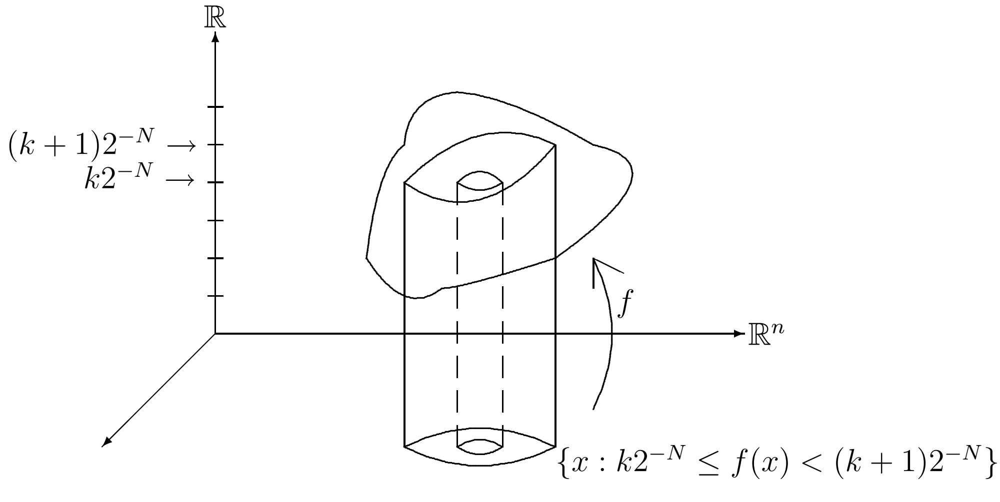
Fact. $f : \mathbb{R}^n \to \mathbb{C}$ is measurable iff for every open set $G \subset \mathbb{C}$, $f^{-1}[G] \in \mathcal{L}$.

Integration

First consider integration of a non-negative function $f : \mathbb{R}^n \to [0, \infty]$, with f measurable. Let N be a positive integer, and define

$$S_N = \sum_{k=0}^{\infty} k2^{-N}\lambda\left(\{x : k2^{-N} \leq f(x) < (k+1)2^{-N}\}\right) + \infty \cdot \lambda(\{x : f(x) = +\infty\}).$$

In the last term on the right-hand-side, we use the convention $\infty \cdot 0 = 0$. The quantity S_N can be regarded as a "lower Lebesgue sum" approximating the volume under the graph of f by subdividing the range space $[0, \infty]$ rather than by subdividing the domain $\mathbb{R}^n$ as in the case of Riemann integration.



Claim. $S_N \le S_{N+1}$.

Proof. We have

$$\begin{aligned}&\{x : k2^{-N} \le f(x) < (k+1)2^{-N}\}\\ &\qquad = \left\{x : k2^{-N} \le f(x) < \left(k+\tfrac{1}{2}\right)2^{-N}\right\} \cup \left\{x : \left(k+\tfrac{1}{2}\right)2^{-N} \le f(x) < (k+1)2^{-N}\right\}\end{aligned}$$

and the union is disjoint. Thus

$$\begin{aligned}&k2^{-N}\lambda(\{x : k2^{-N} \le f(x) < (k+1)2^{-N}\})\\ &\qquad \le k2^{-N}\lambda\left(\left\{x : k2^{-N} \le f(x) < \left(k+\tfrac{1}{2}\right)2^{-N}\right\}\right)\\ &\qquad\qquad + \left(k+\tfrac{1}{2}\right)2^{-N}\lambda\left(\left\{x : \left(k+\tfrac{1}{2}\right)2^{-N} \le f(x) < (k+1)2^{-N}\right\}\right)\end{aligned}$$

and the claim follows after summing and redefining indices. ☐

Definition. The Lebesgue integral of f is defined by:

$$\int_{\mathbb{R}^n} f = \lim_{N\to\infty} S_N.$$

This limit exists (in $[0,\infty]$) by the monotonicity $S_N \le S_{N+1}$.

Other notation for the integral is

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} f d\lambda.$$

General Measurable Functions

Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be measurable. Define

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ 0 & \text{if } f(x) < 0 \end{cases}, \qquad f_-(x) = \begin{cases} 0 & \text{if } f(x) > 0 \\ -f(x) & \text{if } f(x) \le 0 \end{cases}.$$

Then f_+ and f_- are non-negative and measurable, and $(\forall\, x)\ f(x) = f_+(x) - f_-(x)$. The integral of f is only defined if at least one of $\int f_+ < \infty$ or $\int f_- < \infty$ holds, in which case we define

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f_+ - \int_{\mathbb{R}^n} f_-.$$

Definition. A measurable function is called *integrable* if both $\int f_+ < \infty$ and $\int f_- < \infty$. Since $|f| = f_+ + f_-$, this is equivalent to $\int |f| < \infty$.

Properties of the Lebesgue Integral

(We will write $f \in L^1$ to mean f is measurable and $\int |f| < \infty$.)

(1) If $f, g \in L^1$ and $a, b \in \mathbb{R}$, then $af + bg \in L^1$, and $\int (af + bg) = a \int f + b \int g$.

We will write $f = g$ a.e. (almost everywhere) to mean $\lambda\{x : f(x) \neq g(x)\} = 0$.

(2) If $f, g \in L^1$ and $f = g$ a.e., then $\int f = \int g$.

(3) If $f \geq 0$ and $\int f < \infty$, then $f < \infty$ a.e. Thus if $f \in L^1$, then $|f| < \infty$ a.e.

In integration theory, one often identifies two functions if they agree a.e., e.g., $\chi_{\mathbb{Q}} = 0$ a.e.

(4) If $f \geq 0$ and $\int f = 0$, then $f = 0$ a.e. (This is not true if f can be both positive and negative, e.g., $\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = 0$.)

(5) If A is measurable, $\int \chi_A = \lambda(A)$.

Definition. If A is a measurable set and $f : A \to [-\infty, \infty]$ is measurable, then $\int_A f = \int_{\mathbb{R}^n} f \chi_A$.

(6) If A and B are disjoint and $f \chi_{A \cup B} \in L^1$, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

Definition. If $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, and both $\mathcal{R}ef$ and $\mathcal{I}mf \in L^1$, define $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \mathcal{R}ef + i \int_{\mathbb{R}^n} \mathcal{I}mf$.

(7) If $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, then $\mathcal{R}ef$ and $\mathcal{I}mf \in L^1$ iff $|f| \in L^1$. Moreover, $|\int f| \leq \int |f|$.

Comparison of Riemann and Lebesgue integrals

If f is bounded and defined on a bounded set and f is Riemann integrable, then f is Lebesgue integrable and the two integrals are equal.

Theorem. If f is bounded and defined on a bounded set, then f is Riemann integrable iff f is continuous a.e.

Note: The two theories vary in their treatment of infinities (in both domain and range). For example, the improper Riemann integral $\lim_{R \to \infty} \int_0^R \frac{\sin x}{x} dx$ exists and is finite, but $\frac{\sin x}{x}$ is *not* Lebesgue integrable over $[0, \infty)$ since $\int_0^{\infty} \left| \frac{\sin x}{x} \right| dx = \infty$.

Convergence Theorems

Convergence theorems give conditions under which one can interchange a limit with an integral. That is, if $\lim_{k\to\infty} f_k(x) = f(x)$ (maybe only a.e.), where f_k and f are measurable, give conditions which guarantee that $\lim_{k\to\infty} \int f_k = \int f$. This is not true in general:

Examples.

(1) Let $f_k = \chi_{[k,\infty)}$. Then $f_k \geq 0$, $\lim f_k = 0$, and $\int f_k = \infty$, so $\lim \int f_k \neq \int \lim f_k$.

(2) Let $f_k = \chi_{[k,k+1]}$. Then again $\lim f_k = 0$, and $\int f_k = 1$, so $\lim \int f_k \neq \int \lim f_k$.

Monotone Convergence Theorem. (Jones calls this the "Increasing Convergence Theorem".) If $0 \leq f_1 \leq f_2 \leq \cdots$ a.e., $f = \lim f_k$ a.e., and f_k and f are measurable, then $\lim_{k\to\infty} \int f_k = \int f$. Here all the limits are non-negative extended real numbers. Note that $\lim f_k$ exists a.e. by monotonicity.

Fatou's Lemma. If f_k are nonnegative a.e. and measurable, then

$$\int \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int f_k.$$

Lebesgue Dominated Convergence Theorem. Suppose $\{f_k\}$ is a sequence of complex-valued (or extended-real-valued) measurable functions. Assume $\lim_k f_k = f$ a.e., and assume that there exists a "dominating function," i.e., an *integrable* function g such that $|f_k(x)| \leq g(x)$ a.e. Then

$$\int f = \lim_{k\to\infty} \int f_k.$$

A corollary is the

Bounded Convergence Theorem. Let A be a measurable set of finite measure, and suppose $|f_k| \leq M$ in A. Assume $\lim_k f_k$ exists a.e. Then $\lim_k \int_A f_k = \int_A f$. (Apply Dominated Convergence Theorem with $g = M\chi_A$.)

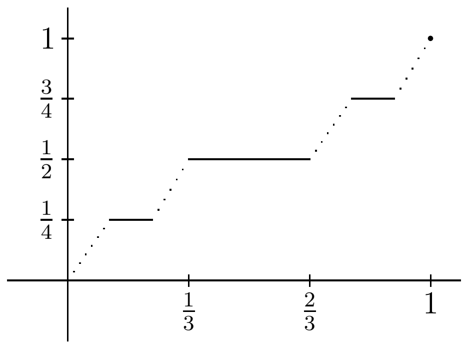
The following result illustrates how Fatou's Lemma can be used together with a dominating sequence to obtain convergence.

Extension of Lebesgue Dominated Convergence Theorem. Suppose $g_k \geq 0$, $g \geq 0$ are all integrable, and $\int g_k \to \int g$, and $g_k \to g$ a.e. Suppose f_k, f are all measurable, $|f_k| \leq g_k$ a.e. (which implies that f_k is integrable), and $f_k \to f$ a.e. (which implies $|f| \leq g$ a.e.). Then $\int |f_k - f| \to 0$ (which implies $\int f_k \to \int f$).

Proof. $|f_k - f| \leq |f_k| + |f| \leq g_k + g$ a.e. Apply Fatou to $g_k + g - |f_k - f|$ (which is ≥ 0 a.e.). Then $\int \liminf(g_k + g - |f_k - f|) \leq \liminf \int (g_k + g - |f_k - f|)$. So $\int 2g \leq \lim \int g_k + \int g - \limsup \int |f_k - f| = 2\int g - \limsup \int |f_k - f|$. Since $\int g < \infty$, $\limsup \int |f_k - f| \leq 0$. Thus $\int |f_k - f| \to 0$. □

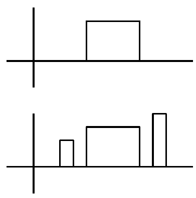
Example — the Cantor Ternary Function.

The Cantor ternary function is a good example in differentiation and integration theory. It is a nondecreasing continuous function $f : [0,1] \to [0,1]$ defined as follows. Let C be the Cantor set. If $x \in C$, say $x = \sum_{k=1}^{\infty} d_k 3^{-k}$ with $d_k \in \{0,2\}$, set $f(x) = \sum_{k=1}^{\infty} \left(\frac{1}{2}d_k\right) 2^{-k}$. Recall that $[0,1]\backslash C$ is the disjoint union of open intervals, the middle thirds which were removed in the construction of C. Define f to be a constant on each of these open intervals, namely $f = \frac{1}{2}$ on $\left(\frac{1}{2}, \frac{2}{3}\right)$, $f = \frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $f = \frac{3}{4}$ on $\left(\frac{7}{9}, \frac{8}{9}\right)$, etc. The general definition is: for $x \in [0,1]$, write $x = \sum_{k=1}^{\infty} d_k 3^{-k}$ where $d_k \in \{0,1,2\}$, let K be the smallest k for which $d_k = 1$, and define $f(x) = 2^{-K} + \sum_{k=1}^{K-1} \left(\frac{1}{2}d_k\right) 2^{-k}$. The graph of f looks like:



Let us calculate $\int_0^1 f(x)dx$ using our convergence theorems. Define a sequence of functions f_k, $k \geq 1$, inductively by:

$$\begin{aligned} f_1 &= \tfrac{1}{2}\chi_{\left(\frac{1}{3},\frac{2}{3}\right)} \\ f_2 &= f_1 + \tfrac{1}{4}\chi_{\left(\frac{1}{9},\frac{2}{9}\right)} + \tfrac{3}{4}\chi_{\left(\frac{7}{9},\frac{8}{9}\right)}, \qquad \text{etc.} \end{aligned}$$



Then each f_k is a simple function, i.e., a finite linear combination of characteristic functions of measurable sets. Note that if $\varphi = \sum_{j=1}^{N} a_j \chi_{A_j}$ is a simple function, where $A_j \in \mathcal{L}$ and $\lambda(A_j) < \infty$, then $\int \varphi = \sum_{j=1}^{N} a_j \lambda(A_j)$. Also, $f_k(x) \to f(x)$ for $x \in [0,1]\backslash C$, and $f_k(x) = 0$ for $x \in C (\forall\, k)$. Since $\lambda(C) = 0$, $f_k \to f$ a.e. on $[0,1]$. So by the MCT or LDCT or BCT, $\int_0^1 f(x)dx = \lim_{k\to\infty} \int_0^1 f_k(x)dx$. Now

$$\int f_k = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3^2}\cdot\frac{1}{2^2}(1+3) + \frac{1}{3^3}\cdot\frac{1}{2^3}(1+3+5+7) + \cdots + \frac{1}{3^k}\cdot\frac{1}{2^k}(1+3+5+\cdots+(2^k-1)).$$

Recall that

$$1 + 3 + 5 + \cdots + (2j - 1) = j^2.$$

So

$$\int f = \lim_k \int f_k = \sum_{m=1}^{\infty} \frac{1}{3^m} \frac{1}{2^m} 2^{2m-2} = \frac{1}{6}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{1}{6}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{2}.$$

(An easier way to see this is to note that $f(1-x) = 1-f(x)$, so $\int_0^1 f(1-x)dx = 1-\int_0^1 f(x)dx$. But changing variables gives $\int_0^1 f(1-x)dx = \int_0^1 f(x)dx$, so $\int_0^1 f(x)dx = \frac{1}{2}$.)

"Multiple Integration" via Iterated Integrals

Suppose $n = m + l$, so $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^l$. For $x \in \mathbb{R}^n$, write $x = (y, z)$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^l$. Then $\int_{\mathbb{R}^n} f d\lambda_n = \int_{\mathbb{R}^n} f(x)d\lambda_n(x) = \int_{\mathbb{R}^n} f(y,z)d\lambda_n(y,z)$. Write dx for $d\lambda_n(x)$, dy for $d\lambda_m(y)$, dz for $d\lambda_l(z)$ (λ_n denotes Lebesgue measure on $\mathbb{R}^n$). Consider the iterated integrals

$$\int_{\mathbb{R}^l}\left[\int_{\mathbb{R}^m} f(y,z)dy\right]dz \qquad \text{and} \qquad \int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^l} f(y,z)dz\right]dy.$$

Questions:

(1) When do these iterated integrals agree?

(2) When are they equal to $\int_{\mathbb{R}^n} f(x)dx$?

There are *two* key theorems, usually used in tandem. The first is Tonelli's Theorem, for non-negative functions.

① **Tonelli's Theorem**. Suppose $f \geq 0$ is measurable on $\mathbb{R}^n$. Then for a.e. $z \in \mathbb{R}^l$, the function $f_z(y) \equiv f(y,z)$ is measurable on $\mathbb{R}^m$ (as a function of y), and

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^l}\left[\int_{\mathbb{R}^m} f(y,z)dy\right]dz.$$

It can happen in Tonelli's Theorem that for some z, the "slice function" f_z is not measurable:

Example. Let $A \subset \mathbb{R}^m$ be non-measurable. Pick $z_0 \in \mathbb{R}^l$ and define $f : \mathbb{R}^n \to \mathbb{R}$ by:

$$f(y,z) = \begin{cases} 0 & (z \neq z_0) \\ \chi_A(y) & (z = z_0) \end{cases}.$$

Then f is measurable on $\mathbb{R}^n$ (since $\lambda_n(\{x : f(x) \neq 0\}) = 0$). But $f_{z_0}(y) = f(y, z_0) = \chi_A(y)$ is not measurable on $\mathbb{R}^m$. However, since the set of z's for which $\int f_z(y)dy$ is undefined has measure zero, the iterated integral still makes sense and is 0.

② **Fubini's Theorem**. Suppose f is integrable on $\mathbb{R}^n$ (i.e., f is measurable and $\int |f| < \infty$). Then for a.e. $z \in \mathbb{R}^l$, the slice functions $f_z(y)$ are integrable on $\mathbb{R}^m$, and

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^m} f(y,z)dy\right] dz.$$

Typically one wants to calculate $\int_{\mathbb{R}^n} f(x)dx$ by doing an iterated integral. One uses Tonelli to verify the hypothesis of Fubini as follows:

(i) Since $|f|$ is non-negative, Tonelli implies that one can calculate $\int_{\mathbb{R}^n} |f|$ by doing either iterated integral. If either one is $< \infty$, then the hypotheses of Fubini have been verified: $\int_{\mathbb{R}^n} |f(x)|dx = \int \left[\int |f(y,z)|dz\right] dy < \infty$.

(ii) Having verified now that $\int_{\mathbb{R}^n} |f| < \infty$, Fubini implies that $\int_{\mathbb{R}^n} f$ can be calculated by doing either iterated integral.

Example. Fubini's Theorem can fail without the hypothesis that $\int |f| < \infty$. Define f on $(0,1) \times (0,1)$ by

$$f(x,y) = \begin{cases} x^{-2} & 0 < y \le x < 1 \\ -y^{-2} & 0 < x < y < 1 \end{cases}.$$

Then $\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \left(\int_0^x x^{-2}dy - \int_x^1 y^{-2}dy\right) dx = \int_0^1 (x^{-1} + 1 - x^{-1})\, dx = 1$. Similarly, $\int_0^1 \int_0^1 f(x,y)dxdy = -1$. Note that by Tonelli,

$$\int_{(0,1)\times(0,1)} |f(x,y)|d\lambda_2(x,y) = \int_0^1 \int_0^1 |f(x,y)|dydx = \int_0^1 \left(x^{-1} + \int_x^1 y^{-2}dy\right) dx = \infty.$$

L^p spaces

$\mathbf{1 \le p < \infty}$. Fix a measurable subset $A \subset \mathbb{R}^n$. Consider measurable functions $f : A \to \mathbb{C}$ for which $\int_A |f|^p < \infty$. Define $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$. On this set of functions, $\|f\|_p$ is only a *seminorm*:

$$\begin{aligned} \|f\|_p &\ge 0 && \text{(but } \|f\|_p = 0 \text{ does not imply } f = 0, \text{ only } f(x) = 0 \text{ a.e.)} \\ \|\alpha f\|_p &= |\alpha| \cdot \|f\|_p && \\ \|f+g\|_p &\le \|f\|_p + \|g\|_p && \text{(Minkowski's Inequality)} \end{aligned}$$

(Note: $\|f\|_p = 0 \Rightarrow \int_A |f|^p = 0 \Rightarrow f = 0$ a.e. on A.) Define an equivalence relation on this set of functions:

$$f \sim g \text{ means } f = g \text{ a.e. on } A.$$

Set $\widetilde{f} = \{g \text{ measurable on } A : f = g \text{ a.e.}\}$ to be the equivalence class of f. Define $\|\widetilde{f}\|_p = \|f\|_p$; this is independent of the choice of representative in $\widetilde{f}$. Define

$$L^p(A) = \{\widetilde{f} : \int_A |f|^p < \infty\}.$$

Then $\|\cdot\|_p$ is a *norm* on $L^p(A)$. We usually abuse notation and write $f \in L^p(A)$ to mean $\widetilde{f} \in L^p(A)$.

Example. We say $f \in L^p(\mathbb{R}^n)$ is "continuous" if $\exists\, g \in L^p(\mathbb{R}^n)$ for which $g : \mathbb{R}^n \to \mathbb{C}$ is continuous and $f = g$ a.e. Equivalently, there exists $g \in \widetilde{f}$ such that g is continuous. In this case, one typically works with the representative g of $\widetilde{f}$ which is continous. This continuous representative is unique since two continuous functions which agree a.e. must be equal everywhere.

$\mathbf{p} = \infty$. Let $A \subset \mathbb{R}^n$ be measurable. Consider "essentially bounded" measurable functions $f : A \to \mathbb{C}$, i.e., for which $\exists\, M < \infty$ so that $|f(x)| \leq M$ a.e. on A. Define

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e. on } A\},$$

the *essential sup* of $|f|$. If $0 < \|f\|_\infty < \infty$, then for each $\epsilon > 0$, $\lambda\{x \in A : |f(x)| > \|f\|_\infty - \epsilon\} > 0$. As above, $\|\cdot\|_\infty$ is a seminorm on the set of essentially bounded measurable functions, and $\|\cdot\|_\infty$ is a norm on

$$L^\infty(A) = \{\widetilde{f} : \|f\|_\infty < \infty\}.$$

Fact. For $f \in L^\infty(A)$, $|f(x)| \leq \|f\|_\infty$ a.e. This is true since

$$\{x : |f(x)| > \|f\|_\infty\} = \bigcup_{m=1}^\infty \{x : |f(x)| > \|f\|_\infty + \frac{1}{m}\},$$

and each of these latter sets has measure 0. So the infimum is attained in the definition of $\|f\|_\infty$.

Fact. $L^\infty(\mathbb{R}^n)$ is *not* separable (i.e., it does not have a countable dense subset).

Example. For each $\alpha \in \mathbb{R}$, let $f_\alpha(x) = \chi_{[\alpha,\alpha+1]}(x)$. For $\alpha \neq \beta$, $\|f_\alpha - f_\beta\|_\infty = 1$. So $\{B_{\frac{1}{3}}(f_\alpha) : \alpha \in \mathbb{R}\}$ is an uncountable collection of disjoint nonempty open subsets in $L^\infty(\mathbb{R})$.

Conjugate Exponents. If $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$ (where $\frac{1}{\infty} \equiv 0$), we say that p and q are *conjugate exponents*. Examples: $\begin{array}{c|cccc} p & 1 & 2 & 3 & \infty \\ q & \infty & 2 & \frac{3}{2} & 1 \end{array}$.

Hölder's Inequality. If $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

(Note: if $\int |fg| < \infty$, also $|\int fg| \leq \int |fg| \leq \|f\|_p \cdot \|g\|_q$.)

Remark. The cases $\left\{ \begin{array}{l} p = 1 \\ q = \infty \end{array} \right.$ and $\left\{ \begin{array}{l} p = \infty \\ q = 1 \end{array} \right.$ are obvious. When $p = 2$, $q = 2$, this is the Cauchy-Schwarz inequality $\int |fg| \leq \|f\|_2 \cdot \|g\|_2$.

Completeness

Theorem. (Riesz-Fischer) Let $A \subset \mathbb{R}^n$ be measurable and $1 \leq p \leq \infty$. Then $L^p(A)$ is complete in the L^p norm $\|\cdot\|_p$.

The completeness of L^p is a crucially important feature of the Lebesgue theory.

Locally L^p Functions

Definition. Let $G \subset \mathbb{R}^n$ be open. Define $L^p_{\text{loc}}(G)$ to be the set of all equivalence classes of measurable functions f on G such that for each compact set $K \subset G$, $f|_K \in L^p(K)$.

There is a metric on L^p_{loc} which makes it a complete metric space (but *not* a Banach space; the metric is not given by a norm). The metric is constructed as follows. Let $K_1, K_2, \ldots$ be a "compact exhaustion" of G, i.e., a sequence of nonempty compact subsets of G with $K_m \subset K^\circ_{m+1}$ (where K°_{m+1} denotes the interior of K_{m+1}), and $\bigcup_{m=1}^{\infty} K_m = G$ (e.g., $K_m = \{x \in G : \text{dist}(x, G^c) \geq \frac{1}{m} \text{ and } |x| \leq m\}$). Then for any compact set $K \subset G$, $K \subset \bigcup_{m=1}^{\infty} K_m \subset \bigcup_{m=1}^{\infty} K^\circ_{m+1}$, so $\exists\, m$ for which $K \subset K_m$. The distance in $L^p_{\text{loc}}(G)$ is

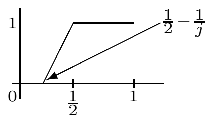
$$d(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_{p,K_m}}{1 + \|f-g\|_{p,K_m}}.$$

It is easy to see that $d(f_j, f) \to 0$ iff $(\forall\, K^{\text{compact}} \subset G)\|f_j - f\|_{p,K} \to 0$.

To see that d is a metric, one uses the fact that if (X, p) is a metric space, and we define $\sigma(x,y) = \frac{p(x,y)}{1+p(x,y)}$, then σ is a metric on X, and (X, p) is uniformly equivalent to (X, σ). To show that σ satisfies the triangle inequality, one uses that $t \mapsto \frac{t}{1+t}$ is increasing on $[0, \infty)$. Note that $\sigma(x,y) < 1$ for all $x, y \in X$.

Continuous Functions not closed in L^p

Let $G \subset \mathbb{R}^n$ be open and *bounded*. Consider $C_b(G)$, the set of bounded continuous functions on G. Clearly $C_b(G) \subset L^p(G)$. But $C_b(G)$ is *not* closed in $L^p(G)$ if $p < \infty$.

Example. Take $G = (0,1)$ and let f_j have graph: 

. Then $\{f_j\}$ is Cauchy in $\|\cdot\|_p$ for $1 \leq p < \infty$. But there is no continuous function f for which $\|f_j - f\|_p \to 0$ as $j \to \infty$.

Facts. Suppose $1 \leq p < \infty$ and $G^{\text{open}} \subset \mathbb{R}^n$.

(1) The set of simple functions (finite linear combinations of characteristic functions of measurable sets) with support in a bounded subset of G is dense in $L^p(G)$.

(2) The set of step functions (finite linear combinations of characteristic functions of rectangles) with support in a bounded subset of G is dense in $L^p(G)$.

(3) $C_c(G)$ is dense in $L^p(G)$, where $C_c(G)$ is the set of continuous functions f whose support $\overline{\{x : f(x) \neq 0\}}$ is a compact subset of G.

(4) $C_c^\infty(G)$ is dense in $L^p(G)$, where $C_c^\infty(G)$ is the set of C^∞ functions whose support is a compact subset of G. (Idea: mollify a given $f \in L^p(G)$. We will discuss this when we talk about convolutions.)

Consequence: For $1 \leq p < \infty$, $L^p(G)$ is separable (e.g., use (2), taking rectangles with rational endpoints and linear combinations with rational coefficients).

Another consequence of the density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ is the continuity of translation. For $f \in L^p(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, define $f_y(x) = f(x-y)$ (translate f by y).

Claim. If $1 \leq p < \infty$, the map $y \mapsto f_y$ from $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$ is uniformly continuous.

Proof. Given $\epsilon > 0$, choose $g \in C_c(\mathbb{R}^n)$ for which $\|g - f\|_p < \frac{\epsilon}{3}$. Let

$$M = \lambda(\{x : g(x) \neq 0\}) < \infty.$$

By uniform continuity of g, $\exists\, \delta > 0$ for which

$$|z - y| < \delta \Rightarrow (\forall\, x)|g_z(x) - g_y(x)| < \frac{\epsilon}{3(2M)^{\frac{1}{p}}}.$$

Then for $|z - y| < \delta$,

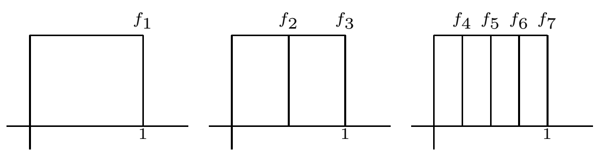
$$\|g_z - g_y\|_p^p = \int |g_z - g_y|^p \leq \lambda(\{x : g_z(x) \neq 0 \text{ or } g_y(x) \neq 0\}) \left(\frac{\epsilon}{3(2M)^{\frac{1}{p}}}\right)^p \leq (2M)\frac{\epsilon^p}{3^p(2M)},$$

i.e., $\|g_z - g_y\|_p \leq \frac{\epsilon}{3}$. Thus $\|f_z - f_y\|_p \leq \|f_z - g_z\|_p + \|g_z - g_y\|_p + \|g_y - f_y\|_p < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. □

L^p convergence and pointwise a.e. convergence

$\mathbf{p = \infty}$. $f_k \to f$ in $L^\infty \Rightarrow$ on the complement of a set of measure 0, $f_k \to f$ uniformly. (Let $A_k = \{x : |f_k(x) - f(x)| > \|f_k - f\|_\infty\}$, and $A = \bigcup_{k=1}^{\infty} A_k$. Since each $\lambda(A_k) = 0$, also $\lambda(A) = 0$. On A^c, $(\forall\, k)|f_k(x) - f(x)| \leq \|f_k - f\|_\infty$, so $f_k \to f$ uniformly on A^c.)

$\mathbf{1 \leq p < \infty}$. Let $A \subset \mathbb{R}^n$ be measurable. Here $f_k \to f$ in $L^p(A)$ (i.e., $\|f_k - f\|_p \to 0$) does not imply that $f_k \to f$ a.e. Example: $A = [0,1]$, $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_3 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{4}]}, \cdots$ etc.



Clearly $\|f_k\|_p \to 0$, so $f_k \to 0$ in L^p, but for no $x \in [0,1]$ does $f_k(x) \to 0$. So L^p convergence for $1 \le p < \infty$ does not imply a.e. convergence. However:

Fact. If $1 \le p < \infty$ and $f_k \to f$ in $L^p(A)$, then $\exists$ a subsequence f_{k_j} for which $f_{k_j} \to f$ a.e. as $j \to \infty$.

Example. Suppose $A \subset \mathbb{R}^n$ is measurable, $1 \le p < \infty$, $f_k, f \in L^p(A)$, and $f_k \to f$ a.e. Question: when does $f_k \to f$ in $L^p(A)$ (i.e. $\|f_k - f\|_p \to 0$)? Answer: In this situation, $f_k \to f$ in $L^p(A)$ iff $\|f_k\|_p \to \|f\|_p$.

Proof.

($\Rightarrow$) If $\|f_k - f\|_p \to 0$, then $|\|f_k\|_p - \|f\|_p| \le \|f_k - f\|_p$, so $\|f_k\|_p \to \|f\|_p$.

($\Leftarrow$) First, observe: If $x, y \ge 0$, then $(x+y)^p \le 2^p(x^p + y^p)$. (Proof: let $z = \max\{x, y\}$; then $(x+y)^p \le (2z)^p = 2^p z^p \le 2^p(x^p + y^p)$.) We will use Fatou's lemma with a "dominating sequence." We have

$$|f_k - f|^p \le (|f_k| + |f|)^p \le 2^p(|f_k|^p + |f|^p).$$

Apply Fatou to $2^p(|f_k|^p + |f|^p) - |f_k - f|^p \ge 0$. By assumption, $\|f_k\|_p \to \|f\|_p$, so $\int |f_k|^p \to \int |f|^p$. We thus get

$$\begin{aligned} \int 2^p(|f|^p + |f|^p) - 0 &\le \liminf \int 2^p(|f_k|^p + |f|^p) - |f_k - f|^p \\ &= \int 2^p(|f|^p + |f|^p) - \limsup \int |f_k - f|^p. \end{aligned}$$

Thus $\limsup \int |f_k - f|^p \le 0$. So $\|f_k - f\|_p^p = \int |f_k - f|^p \to 0$. So $\|f_k - f\|_p \to 0$.

□

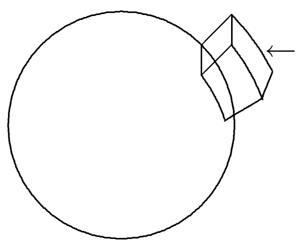
Intuition for growth of functions in $L^p(\mathbb{R}^n)$

Fix n, fix p with $1 \le p < \infty$, and fix $a \in \mathbb{R}$. Define

$$f_1(x) = \frac{1}{|x|^a}\chi_{\{x:|x|<1\}} \qquad f_2(x) = \frac{1}{|x|^a}\chi_{\{x:|x|>1\}}.$$

So f_1 blows up near $x = 0$ for $a > 0$, but vanishes near ∞. And f_2 vanishes near 0 but grows/decays near ∞ at a rate depending on a. To calculate the integrals of powers of f_1 and f_2, use polar coordinates on $\mathbb{R}^n$.

Polar Coordinates in $\mathbb{R}^n$



Here $d\sigma$ is "surface area" measure on S^{n-1}.

Evaluating the integral in polar coordinates,

$$\int_{\mathbb{R}^n} |f_1(x)|^p dx = \int_{S^{n-1}} \left[\int_0^1 \left(\frac{1}{r^a} \right)^p r^{n-1} dr \right] d\sigma = \omega_n \int_0^1 r^{n-ap-1} dr,$$

where $\omega_n = \sigma(S^{n-1})$. This is $< \infty$ iff $n - ap - 1 > -1$, i.e., $a < \frac{n}{p}$. So $f_1 \in L^p(\mathbb{R}^n)$ iff $a < \frac{n}{p}$. Similarly, $f_2 \in L^p(\mathbb{R}^n)$ iff $a > \frac{n}{p}$.

Conclusion. For any $p \neq q$ with $1 \leq p, q \leq \infty$, $L^p(\mathbb{R}^n) \not\subset L^q(\mathbb{R}^n)$.

However, for sets A of finite measure, we have:

Claim. If $\lambda(A) < \infty$ and $1 \leq p < q \leq \infty$, then $L^q(A) \subset L^p(A)$, and

$$\|f\|_p \leq \lambda(A)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Proof. This is obvious when $q = \infty$. So suppose $1 \leq p < q < \infty$. Let $r = \frac{q}{p}$. Then $1 < r < \infty$. Let s be the conjugate exponent to r, so $\frac{1}{r} + \frac{1}{s} = 1$. Then $\frac{1}{s} = 1 - \frac{p}{q} = p\left(\frac{1}{p} - \frac{1}{q}\right)$. By Hölder,

$$\begin{aligned} \|f\|_p^p = \int_A |f|^p &= \int \chi_A |f|^p \leq \|\chi_A\|_s \cdot \| \, |f|^p \|_r = \lambda(A)^{\frac{1}{s}} \left(\int |f|^q \right)^{\frac{p}{q}} \\ &= \lambda(A)^{p\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_q^p. \end{aligned}$$

Now take p^{th} roots. □

Remark. This is in sharp contrast to what happens in l^p. For sequences $\{x_k\}_{k=1}^\infty$, the l^∞ norm is $\|x\|_\infty = \sup_k |x_k|$, and for $1 \leq p < \infty$, the l^p norm is $\|x\|_p = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$.

Claim. For $1 \leq p < q \leq \infty$, $l^p \subset l^q$. In fact $\|x\|_q \leq \|x\|_p$.

Proof. This is obvious when $q = \infty$. So suppose $1 \leq p < q < \infty$. Then

$$\begin{aligned} \|x\|_q^q = \sum_k |x_k|^q &= \sum_k |x_k|^{q-p} |x_k|^p \\ &\leq \|x\|_\infty^{q-p} \sum |x_k|^p \leq \|x\|_p^{q-p} \|x\|_p^p = \|x\|_p^q. \end{aligned}$$

Take q^{th} roots to get $\|x\|_q \leq \|x\|_p$. □