# 4. Assignment 4 (Lecture 17–23 + earlier stuff)

4.1. Suppose that $X$ and $Y$ are two normed spaces and $T : X \to Y$ is a bounded linear operator. Show that if $x_n \to x$ weakly in $X$, then $Tx_n \to Tx$ weakly in $Y$.

4.2. The sequence $(x_n)_{n=1}^{\infty} \subset L^2(\mathbf{R})$ is defined by

$$x_n(t) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{for } n \le t \le 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $(x_n)_{n=1}^{\infty}$ converges weakly to 0 in $L^2(\mathbf{R})$.

(b) Show that the sequence does not converge in $L^2(\mathbf{R})$.

4.3. (a) Assume that $x_n \to x$ weakly in $C([a, b])$, equipped with the $\| \cdot \|_\infty$ norm. Show that $x_n \to x$ pointwise on $[a, b]$.

(b) Let $Y$ denote the subspace of $C([a, b])$ consisting of constant functions. Both spaces are equipped with the $\| \cdot \|_\infty$ norm. Give an example of a bounded linear functional on $Y$, which has infinitely many norm-preserving linear extensions to $C([a, b])$.

4.4. (a) Consider the function $f(x) = \frac{1}{4}x + x^{-1}$, $x \ge 1$. Show that $f$ is a contraction and determine the contraction constant of $f$. Find the (unique) fixed point of $f$.

(b) Is $f(x) = \frac{1}{4}x + x^{-1}$, $x \ge 2$ a contraction?

4.5. Show that any closed vector subspace $Y$ of a normed space $X$ is *weakly sequentially closed*, i.e. if $(x_n)_{n=1}^{\infty}$ is a sequence in $Y$ that converges weakly to $x$, then $x \in Y$.

4.6. Let $X = C([-1, 1])$, equipped with the norm $\| \cdot \|_\infty$. Given $0 \le h \in L^1(-1, 1)$ with $\int_{-1}^{1} h(t)\, dt = 1$, show that the "averaging" functionals $f_n \in X'$, given by

$$f_n(x) = n \int_{-1/n}^{1/n} h(nt)x(t)\, dt, \quad x \in X,$$

converge weak* in $X'$ to the Dirac functional $\delta_0 : x \mapsto x(0)$. What are $f_n$ for $h(t) \equiv \frac{1}{2}$?

4.7. Apply fixed-point iterations to the initial value problem

$$\begin{cases} x'(t) = 1 + x(t)^2, & t \ge 0 \\ x(0) = 0 \end{cases}$$

starting with $x_0(t) = 0$. Verify that the coefficients for $t, t^2, \dots, t^5$ in $x_3(t)$ are the same as in the exact solution to the problem.

4.8. Suppose that $X$ is a Banach space and $S \subset X$ is such that $f(S)$ is bounded for every $f \in X'$. Show that $S$ is bounded.

4.9. Suppose that $H$ is a Hilbert space. Show that $x_n \to x$ in $H$ if and only if $x_n \to x$ weakly in $H$ and $\|x_n\| \to \|x\|$.

## $*$-Problems:

4.10. Suppose that $x \in \ell^\infty$. Show that if the series $\sum_{j=1}^{\infty} x_j y_j$ is convergent for every sequence $y \in \mathbf{c}_0$, then actually $x \in \ell^1$.

4.11. Let $X$ be a Banach space such that its dual $X'$ is reflexive. Assume that $Y \neq X''$ is a closed linear subspace of the bidual $X''$. Show that there exists $f \in X'$ such that $f \neq 0$ but $F(f) = 0$ for all $F \in Y$. If $J : X \to X''$ is the canonical embedding, can $Y = J(X)$? Any conclusion about $X$?

4.12. Suppose that $f \in C([0,1])$ and $\|f\|_\infty \leq 1$. For $0 < \mu < \frac{1}{4}$, show that the equation

$$x(t) + \mu \int_0^t t x^2(s)\, ds = f(t), \quad 0 \leq t \leq 1,$$

has a unique solution $x \in C([0,1])$ such that $\|x\|_\infty \leq 2$.

4.13. Let $(t_n)_{n=1}^{\infty}$ be a sequence in $[0,1]$. Assume that the Dirac functionals

$$\delta_{t_n} : x \mapsto x(t_n)$$

converge weak* to some functional $f$ in the dual $X'$ of $X = (C([0,1]), \|\cdot\|_\infty)$. Show that $f(x) = x(t_0)$ for some $t_0 \in [0,1]$. What does this tell us about the set $F = \{\delta_t : t \in [0,1]\}$ of all Dirac functionals in $X'$?

# 5. Assignment 5 (Lecture 24–30 + earlier stuff)

Some of the problems have several parts and count as normal problems or as $*$-problems, depending on whether you solve the $*$-part or not.

5.1. Investigate if the following operators are open. Justify carefully your claims.

(a) $T : \mathbf{R}^2 \to \mathbf{R}$ defined by $Tx = x_1$ for $x \in \mathbf{R}^2$.

(b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $Tx = (x_1, 0)$ for $x \in \mathbf{R}^2$.

5.2. Show that if $T : X \to Y$ is a closed (not necessarily bounded) linear operator, where $X$ and $Y$ are two normed spaces, then the null space $N(T) := \{x \in X : T(x) = 0\}$ of $T$ is closed.

5.3. Let $T : H \to H$ be a bounded linear operator on a complex Hilbert space $H$.

(a) Show that $N(T) \perp R(T^*)$.

(b) Show that the operator $I + T^*T$ is injective.

5.4. Suppose that $X$ and $Y$ are two normed spaces, $S : X \to Y$ is a closed (not necessarily bounded) linear operator and $T : X \to Y$ is a bounded linear operator. Show that the operator $S + T$ is closed.

5.5. Suppose that $S$ and $T$ are two bounded linear operators on a complex Banach space $X$ that commute, i.e. $ST = TS$. Show that the spectral radii satisfy

$$r(ST) \leq r(S)r(T).$$

Point out where in your arguments you use that $ST = TS$.

5.6. Consider the linear operator $T : \ell^p \to \ell^p$, defined by $Tx = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ...)$ for $x = (x_1, x_2, ...) \in \ell^p$, $p \in [1, \infty]$. This operator is compact (you do not need to prove that). Show that $T$ has no eigenvalues and its spectrum consists of exactly one point.

5.7. Consider the bounded linear operator $T : (C([0, 1]), \|\cdot\|_\infty) \to (C([0, 1]), \|\cdot\|_\infty)$, given by

$$Tx(t) = tx(t), \quad t \in [0, 1].$$

(a) Determine the resolvent set $\rho(T)$ and the resolvent $R_T(\lambda)$ for $\lambda \in \rho(T)$.

(b) Determine the point spectrum $\sigma_p(T)$ and the residual spectrum $\sigma_r(T)$.

5.8. The operator $T : \ell^2 \to \ell^2$ is defined by $Tx = (\alpha_j x_j)_{j=1}^\infty$ for $x = (x_j)_{j=1}^\infty \in \ell^2$, where the sequence $(\alpha_j)_{j=1}^\infty$ is a dense subset of $[0, 1]$.

(a) Show that $T$ is self-adjoint.

(b) Find the point spectrum $\sigma_p(T)$ and show that $[0, 1] \subset \sigma(T)$.

(c) (For a "$*$") Show that the spectrum $\sigma(T) = [0, 1]$ and that $T$ is not compact.

5.9. Consider the integral operator $T : L^2(0, 1) \to L^2(0, 1)$, defined for $x \in L^2(0, 1)$ by

$$Tx(t) = \int_0^1 K(t, s)x(s)\, ds, \quad \text{with kernel } K(t, s) = 3(2\sqrt{ts} + 1).$$

This operator is compact and self-adjoint (you do not need to prove that).

(a) Determine the range $R(T)$, the null space $N(T)$, the spectrum $\sigma(T)$ and the norm $\|T\|$ of $T$.

(b) (For a "$*$") Hilbert–Schmidt diagonalize $T$, i.e. write $Tx$ as a linear combination of $T$'s eigenfunctions. Determine the operator $P := T^2 - 6T + I$ and its range $R(P)$.

5.10. Consider the operator $T : \ell^p \to \ell^p$, $p \in (1,\infty)$, defined for $x = (x_1, x_2, ...) \in \ell^p$ by $Tx = (x_1, \frac{1}{2}(x_2 + x_3), \frac{1}{3}(x_3 + x_4), \frac{1}{4}(x_4 + x_5), ...)$.

(a) Find the operator adjoint $T' : \ell^{p'} \to \ell^{p'}$.

(b) Assuming that $T$ is compact, show that the equation $x = Tx + y$ has for $y = (y_1, y_2, ...) \in \ell^p$ a solution $x \in \ell^p$ if and only if $y_1 = 0$.

(c) (For a "∗") Show that $T$ is compact.

## ∗-Problems:

5.11. Suppose that $X$ and $Y$ are two Banach spaces and $T \in B(X,Y)$ is injective. Consider the operator $T^{-1} : R(T) \to X$. Show that $T^{-1}$ is bounded if and only if the range $R(T)$ is a closed subspace of $Y$.

5.12. (a) In Exercise 5.7, how does the point spectrum $\sigma_p(T)$ change if $T$ is replaced by the operator $Sx(t) = f(t)x(t)$, $t \in [0,1]$, for a fixed $f \in C([0,1])$? Consider e.g. $f(t) = \min\{t, \frac{1}{2}\}$ and try to make a more general conclusion.

(b) In Exercise 5.7, replace $C([0,1])$ by $L^2(0,1)$ and answer the same questions.

# Hints

4.1. You need to show that $f(Tx_n) \to f(Tx)$ for every $f \in Y'$. To which space does the composition $f \circ T$ belong?

4.2. In (a), recall the definition of weak convergence and what is the dual of $L^2(\mathbf{R})$. Justify why various integrals tend to zero. Using the characteristic function $\chi_{[n,2n]}$ in the integral may help but is not necessary. In (b), which is the only possible candidate $x$ for the norm limit in $L^2(\mathbf{R})$? Why?

4.3. In both (a) and (b), consider the functionals $\delta_t(x) := x(t)$ for $a \leq t \leq b$. What are their norms? If $x$ is a constant function, what can you say about $\delta_t(x)$ for different $t$?

4.4. Estimate $|f(x) - f(y)|$. Do not forget to check that $f : X \to X$ for a suitable set $X \subset \mathbf{R}$.

4.5. Recall the definition of weak convergence. Assume that $x \notin Y$ and use one of the corollaries to the Hahn–Banach theorem to obtain a contradiction.

4.6. Use the definition of weak*-convergence.

4.7. Direct calculation. This is more of a Calculus exercise, but it gives you a concrete example of what is going on. It also indicates how some numerical methods may work.

4.8. If $S$ were not bounded, then pick an unbounded sequence $(x_n)_{n=1}^{\infty} \subset S$. Use the canonical embedding $J : X \to X''$ and the Uniform boundedness principle to obtain a contradiction.

4.9. The necessity part is easy and follows from the definition of weak convergence and the continuity of bounded linear functionals. Provide the details. Recall the Riesz representation theorem. For the sufficiency part, expand $\|x - x_n\|^2$ using the definition of the norm in $H$.

4.10. For $n = 1, 2, ...$, consider the linear functionals $f_n(y) = \sum_{j=1}^{n} x_j y_j$ on $\mathbf{c}_0$, where $y = (y_1, y_2, ...) \in \mathbf{c}_0$. Show that $f_n$ are bounded. Apply the Banach–Steinhaus theorem. What is the dual of $\mathbf{c}_0$?

4.11. Recall the definition of the canonical embedding. Apply one of the consequences of the Hahn–Banach theorem to $Y$, together with the reflexivity of $X'$. Remember that $J$ is an isometry, so $J(X)$ is closed in $X''$. This exercise is quite straightforward, but you need to keep track of all the involved objects in suitable spaces and recall the definitions.

4.12. Write the equation as $x = Tx$ for a suitable operator $T$. Show that $T$ is a contraction on a suitable subset $X$ of $C([0, 1])$, equipped with the $\|\cdot\|_\infty$ norm. Remember that you need $T : X \to X$. Use Banach's fixed point theorem.

4.13. Recall that the interval $[0, 1]$ is compact. What do we know about sequences in compact sets? (Alternatively, a special choice of $x(t)$ can be used.) Use that functions in $X$ are continuous. For the last question, find an inspiration in Problem 4.5.

5.1. Are the images $T(G)$ of open sets $G \subset \mathbf{R}^2$ open in the respective target space?

5.2. You need to show that if $x_n \in N(T)$ and $x_n \to x$ in $X$, then $x \in N(T)$. What does this mean in terms of $T(x)$? Use the definition of closed operators.

5.3. In (a), use the definition of Hilbert space adjoint. In (b), assume $(I+T^*T)x = 0$, take the inner product with $x$ and use it to deduce that $\|x\| = 0$.

5.4. You need to show that if $x_n \to x$ in $X$ and $(S+T)(x_n) \to y$ in $Y$, then $y = (S+T)x$. Using the boundedness of $T$, figure out if $Sx_n$ converge (add and subtract $Tx_n$). Use that $S$ is closed.

5.5. Use the Spectral radius formula.

5.6. Recall the definition of eigenvalues. Use the Spectral radius formula and make sure that you use the correct formula for $T^n$. Estimate its norm, do not guess! Or use the Fredholm alternative.

5.7. For $y \in C([0,1])$, solve explicitely the equation $Tx(t) - \lambda x(t) = y(t)$, i.e. find $x(t)$. You need to make sure that you do not divide by zero. For which $\lambda \in \mathbf{C}$ does the solution $x$ belong to $C([0,1])$? Recall the definition of the resolvent set. Justify that the resolvent is bounded from $C([0,1])$ to $C([0,1])$. For $\sigma_p(T)$ and $\sigma_r(T)$, what is the null space $N(T-\lambda I)$ and the range $R(T-\lambda I)$?

5.8. Recall the definition of eigenvalues. Use the density of the sequence $(\alpha_j)_{j=1}^\infty$ and general properties of the spectrum to obtain one inclusion for the spectrum $\sigma(T)$. For the other inclusion, find the inverse (resolvent) $(T-\lambda I)^{-1}$ for $\lambda \in \mathbf{C} \setminus [0,1]$. Justify that it is bounded from $\ell^2$ to $\ell^2$. What do we know about eigenvalues of compact self-adjoint operators? Is this true for $T$?

5.9. To find the eigenvalues, write out what $Tx = \lambda x$ means in this concrete case and observe that for $\lambda \neq 0$, $x$ has to be of special form for this to hold pointwise on the interval $(0,1)$. Compare the coefficients on both sides of the identity $Tx = \lambda x$ to get a system of 2-3 linear equations with $\lambda$ as one of the unknowns.

5.10. Use the Fredholm alternative and study the homogeneous equation $z = T'z$ for $z \in \ell^{p'}$. Do not solve the equation $x = Tx + y$ by hand. For compactness, show that $T$ is a limit of suitable finite rank operators $T_n$.

5.11. Show that if $R(T) \ni y_n = Tx_n \to y \in Y$, then $y = Tx \in R(T)$. What does $Tx_n \to y$ tell us about the sequence $(x_n)_{n=1}^\infty$ when $T^{-1}$ is bounded? Use that $X$ is a Banach space and that $T$ is bounded. The Inverse mapping theorem may also be useful. Verify the assumptions carefully.

5.12. In (a), when can you find $x \in C([0,1])$ such that $(S-\lambda I)x(t) = 0$ for all $t \in [0,1]$? In (b), which useful properties does $T$ have as an operator on the Hilbert space $L^2(0,1)$?