1. Assignment 1 (Lecture 1–5)

1.1. (a) For each point $x = (x_1, x_2)$ in the open square $Q = (-1, 1) \times (-1, 1) \subset \mathbf{R}^2$, find a concrete radius $r_x > 0$ so that the ball (disc) $B(x, r_x) \subset Q$. Use this to write Q as a union of open discs. Compare this with the set $\bigcup_{x\in Q} B(x, r)$ for a fixed $r > 0$.

(b) Let G be a nonempty set in $\mathbf{R}^n$, equipped with the Euclidean metric $|x-y|$. Show that G is open if and only if it is a (possibly uncountable) union of open balls. In fact, the same argument applies in any metric space.

1.2. Suppose that A and B are two subsets of a metric space X. Show that for their closures it holds that

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}. \tag{1}$$

Find disjoint sets $A, B \subset \mathbf{R}$ such that $\overline{A}$ and $\overline{B}$ are not disjoint. What does this tell us about the inclusion (1)?

1.3. Suppose that (X, d) is a metric space and that A is a nonempty subset of X. Show that if x is an accumulation point of A, then every ε-neighbourhood of x contains infinitely many points of A.

1.4. Recall that a metric space (X, d) is totally bounded if

$$\forall\varepsilon > 0 \quad \exists x_1, \dots, x_N \in X \quad X \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon).$$

Now change the order of quantifiers. What does the following property tell us about the space (X, d)? Give a concrete answer $X = \dots$.

$$\exists x_1, \dots, x_N \in X \quad \forall\varepsilon > 0 \quad X \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon).$$

1.5. Recall the definition of completeness and show that the interval $[0, \infty)$ is *not* complete when equipped with the metric

$$d(x, y) = \left| \frac{x}{1+x} - \frac{y}{1+y} \right|, \quad x, y \in [0, \infty).$$

Justify carefully why your sequence is Cauchy and why it fails to converge in $([0, \infty), d)$.

1.6. Let $X = C([-1, 1])$ and put

$$d_1(x, y) = \int_{-1}^{1} |x(t) - y(t)|\, dt \quad \text{and} \quad d_\infty(x, y) = \max_{-1\le t\le 1} |x(t) - y(t)|$$

for $x, y \in X$. Consider the functions

$$x_n(t) = \begin{cases} 1 - 2^n|t|, & \text{if } |t| \leq 2^{-n}, \\ 0, & \text{otherwise} \end{cases} \qquad n = 1, 2, ...$$

and draw a few of them. Show the following:

(a) The sequence $(x_n)_{n=1}^{\infty}$ converges in the metric space (X, d_1).

(b) The sequence $(x_n)_{n=1}^{\infty}$ does *not* converge in (X, d_∞).

1.7. Suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d) and that a subsequence of $(x_n)_{n=1}^{\infty}$ is convergent with limit x. Show that $(x_n)_{n=1}^{\infty}$ is convergent with the same limit.

1.8. (a) Use the definition of compactness (with open covers) to show that the set $K = \{0\} \cup \{2^{-n} : n = 1, 2, ...\}$ is compact.

(b) For every $\varepsilon > 0$, find finitely many balls (i.e. intervals) with radius ε so that they together cover the set $K \setminus \{0\} = \{2^{-n} : n = 1, 2, ...\}$. Which property of $K \setminus \{0\}$ have you just proved?

(c) Show that the set $K \setminus \{0\}$ in (b) is *not* compact by providing an open covering of it, which does not have a finite subcovering.

$*$-Problems:

1.9. Let $X = C([-1, 1])$, equipped with the metrics d_1 and d_∞ defined in Exercise 1.6. Put $V = \{x \in X : |x(t)| < 1 \text{ for all } t \in [-1, 1]\}$.

(a) Show that V is open in (X, d_∞).

(b) Show that the constant function $x(t) = 0$ is *not* an interior point of V with respect to d_1. What does this tell us about the set V in (X, d_1)?

1.10. (a) Assume that $(x_n)_{n=1}^{\infty} \subset \mathbf{c}_0$ and that $x_n \to x$ in ℓ^∞. Show that $x \in \mathbf{c}_0$. Which property of $\mathbf{c}_0$ have you just proved?

(b) Find sequences $x_n = (x_j^{(n)})_{j=1}^{\infty} \in \mathbf{c}_0$ such that $x_j^{(n)} \to 1$ as $n \to \infty$ for all $j = 1, 2, ...$, even though the sequence $x = (1, 1, ...)$ does *not* belong to $\mathbf{c}_0$. What does this tell you about the norm-convergence and the coordinate-wise convergence in $\mathbf{c}_0$?

Note that the superscript $^{(n)}$ just indicates which sequence it belongs to, it does *not* mean the n-th derivative! Another way of writing is in Exercise 1.12. This easily becomes an indexing nightmare but is a good exercise and hard to avoid.

1.11. Suppose that (X, d) is a metric space equipped with the discrete metric d and that sets in $\mathbf{R}$ are equipped with the standard metric.

(a) Which functions from (X, d) to $[-1, 1]$ are continuous?

(b) Which functions from $[-1,1]$ to (X,d) are continuous?

(c) Which functions from $[-1,1] \setminus \{0\}$ to (X,d) are continuous?

1.12. Which of the following statements defines uniform convergence (i.e. in the metric d_∞) of

$$x_n = (x_{n,1}, x_{n,2}, ...) \in l^\infty \quad \text{to} \quad x = (\bar{x}_1, \bar{x}_2, ...) \in l^\infty,$$

as $n \to \infty$? What is described by the remaining statements?

(a) $\forall \varepsilon > 0 \quad \forall j = 1,2,... \quad \exists N \in \mathbf{N} \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(b) $\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall j = 1,2,... \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(c) $\forall j = 1,2,... \quad \exists N \in \mathbf{N} \quad \forall \varepsilon > 0 \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(d) $\exists N \in \mathbf{N} \quad \forall j = 1,2,... \quad \forall \varepsilon > 0 \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

2. Assignment 2 (Lecture 6–12 + earlier stuff)

2.1. Suppose that X is a normed space and Y is a vector (linear) subspace of X. Show that the closure $\overline{Y}$ of Y is a vector (linear) subspace of X, i.e. that $ax + by \in \overline{Y}$ whenever $x, y \in \overline{Y}$ and $a, b \in \mathbf{R}$.

2.2. Suppose that X and Y are two Banach spaces, i.e. normed vector spaces (over the same field) which are complete. They are equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Recall the definitions!

(a) Verify that $\|z\| = \|x\|_X + \|y\|_Y$ is a norm on $X \times Y$, where $z = (x,y) \in X \times Y$.

(b) Show that $X \times Y$ is complete when equipped with this norm.

2.3. Let $\|\cdot\|_p$ denote the norm in ℓ^p, $1 \leq p \leq \infty$.

(a) Show that $\|x\|_\infty \leq \|x\|_p$ if $1 \leq p < \infty$.

(b) Show that $\|x\|_q \leq \|x\|_p$ if $1 \leq p \leq q < \infty$.

(c) Conclude that $\ell^p \subset \ell^q$ for $1 \leq p \leq q \leq \infty$.

2.4. Suppose that the sequences

$$x_n = (x_1^{(n)}, x_2^{(n)}, ...) \to x \text{ in } \ell^p \quad \text{and} \quad y_n = (y_1^{(n)}, y_2^{(n)}, ...) \to y \text{ in } \ell^{p'},$$

where p' is the conjugate exponent to p. Show that $x_n y_n \to xy$ in ℓ^1. Here, the products between sequences are defined coordinate-wise, i.e.

$$x_n y_n = (x_1^{(n)} y_1^{(n)}, x_2^{(n)} y_2^{(n)}, ...).$$

Remember that the superscript $^{(n)}$ just indicates which sequence it belongs to, *not* the n-th derivative!

2.5. Calculate the limit $\lim_{n\to\infty} \int_0^\infty \frac{1}{(x^n + x^{4n})^{1/2n}}\, dx$.

2.6. Calculate the limit $\lim_{n\to\infty} \int_1^\infty \frac{n}{1+x^n}\, dx$.

2.7. Find a continuous function f such that $\int_{\mathbf{R}} |f(x)|\, dx < \infty$ and $f(x) \not\to 0$ as $x \to \infty$.

2.8. Let $(r_n)_{n=1}^\infty$ be an enumeration of all the rational numbers in $(0,1)$ and put

$$f(x) = \sum_{r_n < x} \frac{2^{-n}}{1-r_n}, \qquad x \in [0,1].$$

Show that f is measurable and calculate the integral $\int_0^1 f(x)\, dx$.

∗-Problems:

2.9. Suppose that $f \in L^1(\mathbf{R})$. Show that the integral

$$F(x) = \int_0^x f(t)\, dt, \quad x \in \mathbf{R},$$

is continuous on $\mathbf{R}$, i.e. that $F(x_n) \to F(x)$ when $x_n \to x$.

2.10. Put $f_n(t) = \sin nt,\ t \in (-\pi,\pi),\ n = 1,2,\ldots$, and $S = \{f_n : n = 1,2,\ldots\}$.

(a) Show that S is a bounded subset of $L^2(-\pi,\pi)$.

(b) Show that S is a closed subset of $L^2(-\pi,\pi)$.

(c) Is S compact?

2.11. Find a function $f \notin L^\infty(0,1)$ such that $f \in L^p(0,1)$ for all $1 \le p < \infty$.

2.12. Suppose that $f \in L^\infty(0,1)$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

3. Assignment 3 (Lecture 13–16 + earlier stuff)

Some of the problems have several parts and count as normal problems or as ∗-problems, depending on whether you solve the ∗-part or not.

3.1. For $x \in L^2(-1,1)$ find constants a and b, which minimize the expression

$$\int_{-1}^1 |x(t) - (a+bt)|^2\, dt,$$

that is, find the best approximation of x in $L^2(-1,1)$ by a linear function. What are a and b for $x(t) = t^n$, $n = 2,3,\ldots$?

3.2. Suppose that X is an inner-product space.

(a) If $M_1 \subset M_2 \subset X$, which inclusions hold for $M_1^\perp$ and $M_2^\perp$, and for $(M_1^\perp)^\perp$ and $(M_2^\perp)^\perp$?

(b) Let $(x_n)_{n=1}^\infty$ be a sequence in X such that $x_n \to x$. Show that if $x_n \perp y$ for some $y \in X$ and all $n = 1, 2, ...$, then $x \perp y$. Which property of

$$M^\perp = \{x \in X : x \perp y \text{ for all } y \in M\}$$

follows from this for an arbitrary set $M \subset X$?

3.3. Let $S = \{x \in L^2(-1,1) : \int_0^1 x(t)\, dt = 1\}$. Show that S contains a unique element with minimal L^2-norm and find this element.

3.4. (a) Let α and β be two fixed numbers. Show that

$$g(x) = \alpha x(0) + \beta \int_0^1 x(t^2)\, dt, \quad x \in C([0,1])$$

is a bounded linear *functional on* the space $X := (C([0,1]), \|\cdot\|_\infty)$, i.e. from X to $\mathbf{R}$ (or $\mathbf{C}$).

(b) Show that

$$T : x \mapsto t^2 \int_a^t x(s)\, ds, \quad x \in C([a,b])$$

is a bounded linear *operator on* the space $X := (C([a,b]), \|\cdot\|_\infty)$, i.e. from X to X. Give a concrete (not necessarily the best, but correct!) estimate of the norm $\|T\|$ in terms of a and b.

3.5. Let $T : X \to Y$ be an invertible bounded linear operator between normed spaces X and Y. Show that $\|T^{-1}\| \geq \|T\|^{-1}$. Calculate $\|T\|$ and $\|T^{-1}\|$ for $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1, 2x_2)$. What does this tell us about the inequality $\|T^{-1}\| \geq \|T\|^{-1}$?

3.6. Let L denote the left-shift operator on ℓ^2, i.e. $L(x_1, x_2, ...) = (x_2, x_3, ...)$.

(a) Is L injective, i.e. does $Lx = Ly$ imply $x = y$?

(b) Is L surjective, i.e. $L(\ell^2) = \ell^2$?

(c) Calculate $\|L^n\|$, $\lim_{n\to\infty} \|L^n\|$ and $\lim_{n\to\infty} \|L^n x\|_2$, where $x \in \ell^2$ and L^n denotes the composition of L with itself $n-1$ times.

(d) What does (c) say about the pointwise and the norm convergence of the operators L^n, as $n \to \infty$?

3.7. On $X = \{x \in C^1([0,2]) : x(0) = 0\}$, consider the linear functional $f(x) = x(2)$.

(a) Show that f is bounded with respect to the norm $\|x\|_* := \left(\int_0^2 |x'|^2\, dt\right)^{1/2}$.

(b) Write f as $f(x) = (x, z)_*$ for a suitable function $z \in X$, where $(\cdot, \cdot)_*$ is the inner product associated with the above norm $\| \cdot \|_*$.

(c) (For a "$*$") Calculate the norm of f with respect to $\| \cdot \|_*$ and with respect to $\| \cdot \|_\infty$.

3.8. Let $Z = \{x \in L^2(-1, 1) : \int_{-1}^{1} x(t)\, dt = 0\}$.

(a) Show that Z is a closed linear subspace of $L^2(-1, 1)$.

(b) (For a "$*$") Show that $Y := Z^\perp$ consists exactly of all constant functions. Given $x \in L^2(-1, 1)$, determine the orthogonal decomposition $x = x_Y + x_Z$, where $x_Y \in Y$ and $x_Z \in Z$.

$*$-Problems:

3.9. Calculate the norm $\|T\|$ of the linear functional T on $(C([-1, 1]), \| \cdot \|_\infty)$ defined by

$$Tx = \int_0^1 x(t)\, dt - \int_{-1}^{0} x(t)\, dt, \quad x \in C([-1, 1]).$$

Is there $x \in C([-1, 1])$ such that $\|x\|_\infty = 1$ and $Tx = \|T\|$?

3.10. Show that if f is a bounded linear functional on $\mathbf{c}_0$ then there is $y \in \ell^1$ such that

$$f(x) = \sum_{j=1}^{\infty} x_j y_j \quad \text{for all } x \in \mathbf{c}_0.$$

3.11. Suppose that H is a Hilbert space and let $(e_n)_{n=1}^{\infty}$ be an infinite ON-sequence in H. Put $Y = \text{span}\{e_1, e_2, ...\}$, i.e. Y consists of all *finite* linear combinations of the e_n's.

(a) Show that if $x = \sum_{n=1}^{\infty} (x, e_n) e_n$ then $x \in \overline{Y}$.

(b) Show that if $x \in \overline{Y}$ then $x = \sum_{n=1}^{\infty} (x, e_n) e_n$.

Hints

1.1. Remember that every set is the union of all of its points. In (b), remember that "if and only if" is an equivalence, so you need to prove two implications. Which ones? What do we know about unions of open sets?

1.2. Use some of the characterizations of the closure $\overline{A}$ that we had in the lectures.

1.3. Assume that the conclusion is not true and deduce that x is then not an accumulation point of A. Do not ignore the condition $y \neq x$ in the definition of accumulation points.

1.4. A picture may help. For an arbitrary $x \in X$, what can you say about $d(x, x_j)$ for $j = 1, \ldots, N$?

1.5. Assume that your sequence converges in the metric d to some $x \in [0, \infty)$. What can you say about x? Does it exist? Use only points from the metric space $[0, \infty)$ and convergence with respect to the metric d, i.e. *not* the usual Euclidean metric on $\mathbf{R}$.

1.6. (a) A good candidate for the limit is the constant function $x(t) = 0$. For a simple integration, recall that integral corresponds to an area.

(b) Is $(x_n)_{n=1}^{\infty}$ a Cauchy sequence with respect to d_∞?

1.7. Use the definition of convergence and of Cauchy sequences.

1.8. Pictures may help.

1.9. (a) For example, you can show that every $x \in V$ is an interior point of V with respect to d_∞. You can even find a concrete $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V$.

(b) What is $d_1(0, y)$ for a function $y \in (X, d_1)$? Can you make this small while $y \in V$?

1.10. Look at the proof in the lectures showing that uniform convergence preserves continuity. Do not mix the elements $x_j^{(n)}$ in each sequence x_n with the sequences themselves as members in a converging sequence $x_n \to x$. Keep tract of the indices.

1.11. (a) If $x_n \to x$ in (X, d), what can we say about x_n for sufficiently large n? Use this and some of the equivalent characterizations of continuity that we had in the lectures. In (b) and (c), considering $f^{-1}(\{x\})$ for two different x's may also be useful. What is the difference between $[-1, 1]$ and $[-1, 1] \setminus \{0\}$ in this situation?

2.1. Recall the definition of vector spaces and that $x \in \overline{Y}$ if and only if there exist $x_n \in Y$ such that $\|x_n - x\| \to 0$.

2.2. Write out carefully the involved norms and do careful but rather straightforward estimates. In (b), you need to show that for every Cauchy sequence (x_n, y_n) in $X \times Y$ there is $(x, y) \in X \times Y$ such that $(x_n, y_n) \to (x, y)$. Take such a Cauchy sequence. How can you find $x \in X$ and $y \in Y$? Remember that X and Y are complete. Justify all your claims in the right order.

2.3. In (a), compare $|x_j|^p$ and $\|x\|_p$. If you use $\max_j |x_j|^p$, explain why it exists. Using $\sup_j$ is easier and safer. In (b), prove the inequality first under the assumption that $\|x\|_p = 1$. Then, use the homogeneity of the norm, together with the fact that $y := x/\|x\|_p$ has $\|y\|_p = 1$. In (c) explain why $x \in \ell^p$ implies $x \in \ell^q$. You have done most of the work already.

2.4. Somewhere you will need to use Hölder's inequality (twice, actually). It is useful to write it as $\|xy\|_1 \leq \|x\|_p \|y\|_{p'}$. What does it mean that $x_n y_n \to xy$ in $\| \cdot \|_1$? Inserting $\pm x_n y$ and the triangle inequality also help. Recall that converging sequences are bounded.

2.5. We have convergence theorems for the Lebesgue integral. Split the integral at $x = 1$ into two integrals. In each of them, which nice *integrable* function dominates the integrand?

2.6. We have convergence theorems for the Lebesgue integral. Begin by changing variables and remember that you are taking a limit as $n \to \infty$. So, how important are the first $n = 1, 2, 3$ for the limit?

2.7. Recall that $f(x) \not\to 0$ means that either $\lim_{x\to\infty} f(x) \to c \neq 0$ or that

$$\limsup_{x\to\infty} f(x) > \liminf_{x\to\infty} f(x).$$

Exclude one of the possibilities and note that f will not be monotone.

2.8. Write $f(x)$ as a limit of partial sums with $\chi_{(r_n,1)}$. How can we make new measurable functions out of old ones? *Warning*: All rational numbers in $[0, 1]$ *cannot* be ordered in a *monotone* sequence, so do *not* use that.

2.9. Write $F(x_n) - F(x)$ as an integral. We have convergence theorems for the Lebesgue integral. *Warning*: The convergence $x_n \to x$ need not be monotone.

2.10. Which are the converging sequences in S? What is $\|f_n - f_m\|_2$? Use some of the characterizations of closed and compact sets that we had in the lectures.

2.11. For which $\alpha < 0$ is the power $x^\alpha \in L^p(0, 1)$? Can you find $0 \leq f \notin L^\infty(0, 1)$ which is majorized at 0 by all such powers?

2.12. One inequality is easy. For the other inequality and a fixed $\varepsilon > 0$ consider the set $A := \{x : |f| \geq (1 - \varepsilon)\|f\|_\infty\}$. Estimate $\|f\|_p$ from below using A.

3.1. Use orthogonal projections and do not forget to normalize your ON-basis properly.

3.2. In (a), use the definition of $M^\perp$. Remember that M_1 and M_2 need not be linear subspaces. In (b), try $x_n \in M^\perp$ and $y \in M$.

3.3. Use the Cauchy–Schwarz inequality with a suitable function $y \in L^2(-1,1)$ in the integral $\int_0^1 x(t)\,dt$ to get a lower bound for $\|x\|_2$ for any $x \in S$. When do we have equality in the Cauchy–Schwarz inequality?

3.4. Recall the definition of bounded linear operators and functionals. Do not abuse absolute values. Warning: a and b can be negative or zero!

3.5. Use e.g. the supremum formula for calculating $\|T\|$ and $\|T^{-1}\|$. You can also use that $T^{-1}T = I$.

3.6. To calculate the norms, you need estimates both from above and from below.

3.7. Rewrite f as an integral. What do we know about bounded linear functionals on Hilbert spaces?

3.8. In (a), you can use the inner product. Consider e.g. $x_n \in Z$ such that $x_n \to x$ in $L^2(-1,1)$ and show that $x \in Z$. Justify carefully convergence of the integrals. In (b), write Z as an orthogonal complement.

3.9. First, estimate $|Tx|$ using $\|x\|_\infty$. Then find $x_n \in C([-1,1])$ with $\|x_n\|_\infty = 1$ such that $Tx_n \to \|T\|$.

3.10. See the proof from the lecture showing that the dual of ℓ^p is $\ell^{p'}$.

3.11. (a) Consider the partial sums in the above infinite series.

(b) Consider $z = x - \sum_{n=1}^{\infty}(x, e_n)e_n$ and show that $z \perp Y$. Conclusion?