

4. Assignment 4 (Lecture 18–23 + earlier stuff)

- 4.1. (a) Consider the function $f(x) = \frac{1}{4}x + x^{-1}$, $x \geq 1$. Show that f is a contraction and determine the contraction constant of f . Find the (unique) fixed point of f .

(b) Is $f(x) = \frac{1}{4}x + x^{-1}$, $x \geq 2$ a contraction?

- 4.2. Assume that $x_n \rightarrow x$ weakly in $C([a, b])$, equipped with the $\|\cdot\|_\infty$ norm. Show that $x_n \rightarrow x$ pointwise on $[a, b]$. Conversely, find $y_n \in C([0, 1])$ such that $y_n \rightarrow 0$ pointwise on $[0, 1]$ but not weakly in $C([0, 1])$.

- 4.3. The sequence $(x_n)_{n=1}^\infty \subset L^2(\mathbf{R})$ is defined by

$$x_n(t) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{for } n \leq t \leq 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $(x_n)_{n=1}^\infty$ converges weakly to 0 in $L^2(\mathbf{R})$.

(b) Show that the sequence does not converge in $L^2(\mathbf{R})$ (to any $x \in L^2(\mathbf{R})$).

- 4.4. Let Y denote the subspace of $X := L^1(-1, 1)$ consisting of constant functions. Both spaces are equipped with the $\|\cdot\|_1$ norm. Calculate the norm $\|f\|$ of the bounded linear functional

$$f(x) = \frac{1}{2} \int_0^1 x(t) dt \quad \text{on } Y$$

and find infinitely many norm-preserving linear extensions of f to X .

- 4.5. Apply fixed-point iterations to the initial value problem

$$\begin{cases} x'(t) = 1 + x(t)^2, & t \geq 0 \\ x(0) = 0 \end{cases}$$

starting with $x_0(t) = 0$. Verify that the coefficients for $t, t^2, \dots, t^5$ in $x_3(t)$ are the same as in the exact solution to the problem.

- 4.6. Show that any closed vector subspace Y of a normed space X is *weakly sequentially closed*, i.e. if $(x_n)_{n=1}^\infty$ is a sequence in Y that converges weakly to x , then $x \in Y$. Give an example of a sequence in a Hilbert space H , that shows that the unit sphere $S := \{x \in H : \|x\| = 1\}$ (which is a closed set) is not weakly sequentially closed.

- 4.7. Let $X = C([-1, 1])$, equipped with the norm $\|\cdot\|_\infty$. Given $0 \leq h \in L^1(-1, 1)$ with $\int_{-1}^1 h(t) dt = 1$, show that the “averaging” functionals $f_n \in X'$, given by

$$f_n(x) = n \int_{-1/n}^{1/n} h(nt)x(t) dt, \quad x \in X,$$

converge weak* in X' to the Dirac functional $\delta_0 : x \mapsto x(0)$. What are f_n for $h(t) \equiv \frac{1}{2}$? (Give a concrete explicit answer. A picture may help.)

- 4.8. Suppose that X is a normed space and $S \subset X$ is such that $f(S)$ is bounded for every $f \in X'$. Show that S is bounded.
- 4.9. Suppose that H is a Hilbert space. Show that $x_n \rightarrow x$ in H if and only if $x_n \rightarrow x$ weakly in H and $\|x_n\| \rightarrow \|x\|$.

***-Problems:**

- 4.10. Suppose that $x \in \ell^\infty$. Show that if the series $\sum_{j=1}^\infty x_j y_j$ is convergent for every sequence $y \in \mathbf{c}_0$, then actually $x \in \ell^1$.
- 4.11. Let X be a Banach space such that its dual X' is reflexive. Assume that $Y \neq X''$ is a closed linear subspace of the bidual X'' . Show that there exists $f \in X'$ such that $f \neq 0$ but $F(f) = 0$ for all $F \in Y$. If $J : X \rightarrow X''$ is the canonical embedding, can $Y = J(X)$? Any conclusion about X ?
- 4.12. Suppose that $f \in C([0, 1])$ and $\|f\|_\infty \leq 1$. For $0 < \mu < \frac{1}{4}$, show that the equation

$$x(t) + \mu \int_0^t tx^2(s) ds = f(t), \quad 0 \leq t \leq 1,$$

has a unique solution $x \in C([0, 1])$ such that $\|x\|_\infty \leq 2$.

- 4.13. Let $(t_n)_{n=1}^\infty$ be a sequence in $[0, 1]$ and consider the Dirac functionals

$$\delta_{t_n} : x \mapsto x(t_n)$$

in the dual X' of $X = (C([0, 1]), \|\cdot\|_\infty)$. Show that a subsequence $\{\delta_{t_{n_k}}\}_{k=1}^\infty$ of $\{\delta_{t_n}\}_{n=1}^\infty$ converges weak* to some functional $f \in X'$. What is f ? What does this tell us about the set

$$F = \{\delta_t : t \in [0, 1]\}$$

of all Dirac functionals in X' ? Does the subsequence $\{\delta_{t_{n_k}}\}_{k=1}^\infty$ converge in the norm $\|\cdot\|_{X'}$? Is F compact?

5. Assignment 5 (Lecture 24–30 + earlier stuff)

Some of the problems have several parts and count as normal problems or as *-problems, depending on whether you solve the *-part or not.

5.1. Investigate if the following operators are open. Justify carefully your claims.

(a) $T : \mathbf{R}^2 \rightarrow \mathbf{R}$ defined by $Tx = x_1$ for $x \in \mathbf{R}^2$.

(b) $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $Tx = (x_1, 0)$ for $x \in \mathbf{R}^2$.

5.2. Let $T : H \rightarrow H$ be a bounded linear operator on a complex Hilbert space H . Show that the operator $I + T^*T$ is injective.

5.3. Suppose that X and Y are two normed spaces, $S : X \rightarrow Y$ is a closed (not necessarily bounded) linear operator and $T : X \rightarrow Y$ is a bounded linear operator. Show that the operator $S + T$ is closed.

5.4. Consider the linear operator $T : \ell^p \rightarrow \ell^p$, defined by $Tx = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$ for $x = (x_1, x_2, \dots) \in \ell^p$, $p \in [1, \infty]$. Show that T has no eigenvalues and its spectrum consists of exactly one point. To which part of the spectrum does this point belong?

5.5. Consider the operator $T : \ell^p \rightarrow \ell^p$, $p \in (1, \infty)$, defined for $x = (x_1, x_2, \dots) \in \ell^p$ by $Tx = (x_1, \frac{1}{2}(x_2 + x_3), \frac{1}{3}(x_3 + x_4), \frac{1}{4}(x_4 + x_5), \dots)$.

(a) Find the operator adjoint $T' : \ell^{p'} \rightarrow \ell^{p'}$.

(b) Assuming that T is compact (you do not need to prove that), find out for which $y = (y_1, y_2, \dots) \in \ell^p$ does the equation $x = Tx + y$ have a solution $x \in \ell^p$.

5.6. Let X be a complete metric space without isolated points. (Such spaces are called perfect.) For $x \in X$, is the set $X \setminus \{x\}$ dense in X ? Which other useful property does it have? Can you use this and the Baire theorem to show that X must be uncountable?

5.7. Consider the integral operator $T : L^2(0, 1) \rightarrow L^2(0, 1)$, defined for $x \in L^2(0, 1)$ by

$$Tx(t) = \int_0^1 K(t, s)x(s) ds, \quad \text{with kernel } K(t, s) = 3(2\sqrt{ts} + 1).$$

This operator is compact and self-adjoint (you do not need to prove that). Determine the range $R(T)$, the null space $N(T)$, the spectrum $\sigma(T)$ and the norm $\|T\|$ of T .

5.8. The operator $T : \ell^2 \rightarrow \ell^2$ is defined by $Tx = (\alpha_j x_j)_{j=1}^\infty$ for $x = (x_j)_{j=1}^\infty \in \ell^2$, where the sequence $(\alpha_j)_{j=1}^\infty$ forms a dense subset of $[0, 1]$.

(a) Show that T is self-adjoint.

(b) Find the point spectrum $\sigma_p(T)$ and show that $[0, 1] \subset \sigma(T)$.

- (c) (For a “*”) Show that the spectrum $\sigma(T) = [0, 1]$ and that T is not compact.
- 5.9. For $X = C([0, 1], \|\cdot\|_\infty)$, consider the bounded linear operator $T : X \rightarrow X$, given by
- $$Tx(t) = tx(t), \quad t \in [0, 1].$$
- (a) Determine the resolvent set $\rho(T)$ and the resolvent $R_T(\lambda)$ for $\lambda \in \rho(T)$.
- (b) Determine the point spectrum $\sigma_p(T)$.
- (c) (For a “*”) Determine the residual spectrum $\sigma_r(T)$ and the continuous spectrum $\sigma_c(T)$. What would the residual spectrum be if $X = C([0, 1])$ was replaced by $X = L^2(0, 1)$?

*-Problems:

- 5.10. Consider the integral operator $T : L^2(0, 1) \rightarrow L^2(0, 1)$ from Exercise 5.7. You can use the results from that exercise (so you need to solve it first).
- (a) Write Tx as a linear combination of T 's eigenfunctions. (You do not need to calculate the eigenfunctions.) Can you write $x \in L^2(0, 1)$ using these eigenfunctions? Determine the operator $P := T^2 - 6T + I$ and its range $R(P)$.
- (b) For which $f \in L^2(0, 1)$ does the equation $x = Tx + f$ have a solution?
- 5.11. Suppose that X and Y are two Banach spaces and $T \in B(X, Y)$ is injective. Consider the operator $T^{-1} : R(T) \rightarrow X$. Show that T^{-1} is bounded if and only if the range $R(T)$ is a closed subspace of Y .
- 5.12. In Exercise 5.9, how does the point spectrum $\sigma_p(T)$ change if T is replaced by the operator $Sx(t) = f(t)x(t)$, $t \in [0, 1]$, for a fixed function $f \in C([0, 1])$? Consider e.g. $f(t) = \min\{t, \frac{1}{2}\}$ and make a more general conclusion.

Hints

- 4.1. Estimate $|f(x) - f(y)|$. Do not forget to check that $f : X \rightarrow X$ for a suitable set $X \subset \mathbf{R}$.
- 4.2. Which functionals on $C([0, 1])$ could give the pointwise convergence? For the second part, you need a bounded linear functional f on $C([0, 1])$ such that $f(y_n) \not\rightarrow 0$. How do typical bounded linear functionals on $C([0, 1])$ look like? Which sequences of pointwise converging functions did we have in the lectures?
- 4.3. In (a), recall the definition of weak convergence and what is the dual of $L^2(\mathbf{R})$. Justify why various integrals tend to zero. Using the characteristic function $\chi_{[n, 2n]}$ in the integral, together with a convergence theorem for integrals, may help but is not necessary. In (b), which is the only possible candidate x for the norm limit in $L^2(\mathbf{R})$? Why?
- 4.4. If $x(t) \equiv c$ is a constant function, what is $f(x)$? Are there other functionals g such that $g(x) = f(x)$ for constant x ? Check that the norm is really preserved, i.e. $\|g\| = \|f\|$.
- 4.5. Direct calculation. This is more of a Calculus exercise, but it gives you a concrete example of what is going on. It also indicates how some numerical methods may work.
- 4.6. Recall the definition of weak convergence. Assume that $x \notin Y$ and use one of the corollaries to the Hahn–Banach theorem to obtain a contradiction. For the last part, what property should that sequence have? We had such an example in several lectures.
- 4.7. Use the definition of weak*-convergence.
- 4.8. If S were not bounded, then pick an unbounded sequence $(x_n)_{n=1}^\infty \subset S$. Use the canonical embedding $J : X \rightarrow X''$ and the Uniform boundedness principle to obtain a contradiction.
- 4.9. The necessity part is easy and follows from the definition of weak convergence and the continuity of bounded linear functionals. Provide the details. Recall the Riesz representation theorem. For the sufficiency part, expand $\|x - x_n\|^2$ using the definition of the norm in H .
- 4.10. For $n = 1, 2, \dots$, consider the linear functionals $f_n(y) = \sum_{j=1}^n x_j y_j$ on $\mathbf{c}_0$, where $y = (y_1, y_2, \dots) \in \mathbf{c}_0$. What is the dual of $\mathbf{c}_0$? Which theorem can you then apply?
- 4.11. Recall the definition of the canonical embedding. Apply one of the consequences of the Hahn–Banach theorem to Y , together with the reflexivity of X' . Remember that J is an isometry, so $J(X)$ is closed in X'' . This exercise is quite straightforward, but you need to keep track of all the involved objects in suitable spaces and recall the definitions.

- 4.12. Write the equation as $x = Tx$ for a suitable operator T . Show that T is a contraction on a suitable subset X of $C([0, 1])$, equipped with the $\|\cdot\|_\infty$ norm. Remember that you need $T : X \rightarrow X$.
- 4.13. Recall that the interval $[0, 1]$ is compact. What do we know about sequences in compact sets? Use that functions in X are continuous. A special choice of $x(t)$ can also be used. You can also get inspiration from Exercise 2.9.
- 5.1. Are the images $T(G)$ of open sets $G \subset \mathbf{R}^2$ open in the respective target space? (A suitable theorem can also be used but it is an overkill.)
- 5.2. Assume that $(I + T^*T)x = 0$ and take the inner product with x .
- 5.3. You need to show that if $x_n \rightarrow x$ in X and $(S + T)(x_n) \rightarrow y$ in Y , then $y = (S + T)x$. Using the boundedness of T , figure out if Sx_n converge. Use that S is closed. For a good logical structure of the proof it may be convenient to use the boxes and implications as in Exercise 2.2.
- 5.4. Recall the definition of eigenvalues. Use the Spectral radius formula and make sure that you use the correct formula for T^n . Estimate its norm, do not guess!
- 5.5. Use the Fredholm alternative and study the homogeneous equation $z = T'z$ for $z \in \ell^{p'}$. Do not solve the equation $x = Tx + y$ by hand.
- 5.6. Assume that $X = \bigcup_{n=1}^\infty \{x_n\}$ is a countable union of singletons and get a contradiction.
- 5.7. To find the eigenvalues, write out what $Tx = \lambda x$ means in this concrete case and observe that for $\lambda \neq 0$, x has to be of special form for this to hold pointwise on the interval $(0, 1)$. Compare the coefficients on both sides of the identity $Tx = \lambda x$ to get a system of 2-3 linear equations with λ as one of the unknowns.
- 5.8. Recall the definition of eigenvalues. Use the density of the sequence $(\alpha_j)_{j=1}^\infty$ and general properties of the spectrum to obtain one inclusion for the spectrum $\sigma(T)$. For the other inclusion, find the inverse (resolvent) $(T - \lambda I)^{-1}$ for $\lambda \in \mathbf{C} \setminus [0, 1]$. Justify that it is bounded from ℓ^2 to ℓ^2 . What do we know about eigenvalues of compact self-adjoint operators? Is this true for T ?
- 5.9. For $y \in C([0, 1])$, solve explicitly the equation $Tx(t) - \lambda x(t) = y(t)$, i.e. find $x(t)$. You need to make sure that you do not divide by zero. For which $\lambda \in \mathbf{C}$ does the solution x belong to $C([0, 1])$? Recall the definition of the resolvent set. Justify that the resolvent is bounded from $C([0, 1])$ to $C([0, 1])$. For $\sigma_p(T)$ and $\sigma_r(T)$, what is the null space $N(T - \lambda I)$ and the range $R(T - \lambda I)$? For $X = L^2(0, 1)$, which useful property does T have as an operator on the Hilbert space $L^2(0, 1)$?
- 5.10. Three mathematicians, Hilbert, Schmidt and Fredholm, can help.
- 5.11. You need to prove two implications. For one of them, show that if $R(T) \ni y_n = Tx_n \rightarrow y \in Y$, then $y = Tx \in R(T)$. What does $Tx_n \rightarrow y$ tell us about the sequence $(x_n)_{n=1}^\infty$? For a good logical structure, you can use the boxes and

implications as in Exercise 2.2. The Inverse mapping theorem or the Closed graph theorem may also be useful. Verify their assumptions carefully!

- 5.12. When can you find $0 \neq x \in C([0, 1])$ such that $(S - \lambda I)x(t) = 0$ for all $t \in [0, 1]$?