YOU CAN USE THE HINTS TO THE EXERCISES AT THE END OF THE FILE

Note that especially Exercises 1.8 and 2.2 have long texts, so do not give up reading!

1. Assignment 1 (Lecture 1–5)

1.1. (a) For each point $x = (x_1, x_2)$ in the open square $Q = (-1, 1) \times (-1, 1) \subset \mathbf{R}^2$, find a concrete (not necessarily best) radius $r_x > 0$ so that the ball (disc) $B(x, r_x) \subset Q$. Use this to write Q as a union of open discs. Compare this with the set $\bigcup_{x \in Q} B(x, r)$ for a fixed $r > 0$ and draw a picture of such a set.

(b) Let G be a nonempty set in $\mathbf{R}^n$, equipped with the Euclidean metric $|x-y|$. Show that G is open if and only if it is a (possibly uncountable) union of open balls. In fact, the same argument applies in any metric space.

1.2. Suppose that $A_1 \subset A_2$ are two subsets of a metric space X and x is an accumulation point of A_1. Show that it is an accumulation point of A_2. Does this imply any of the following inclusions for closures of sets $A, B \subset X$?

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B} \qquad \text{or} \qquad \overline{A} \cap \overline{B} \subset \overline{A \cap B}? \tag{1}$$

Is the other inclusion true? Proof or counterexample!

1.3. Suppose that (X, d) is a metric space and that A is a nonempty subset of X. Show that if x is an accumulation point of A, then every ε-neighbourhood of x contains infinitely many points of A.

1.4. Recall that a metric space (X, d) is totally bounded if

$$\forall \varepsilon > 0 \quad \exists x_1, \dots, x_N \in X \quad X \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon).$$

Now change the order of quantifiers. What does the following property tell us about the space (X, d)?

$$\exists x_1, \dots, x_N \in X \quad \forall \varepsilon > 0 \quad X \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon).$$

Give a concrete answer $X = \dots$ and argue why it is so.

1.5. (a) Consider the sequence $x_n = n$ in $[0, \infty)$, equipped with the metric

$$d(x, y) = \left| \frac{x}{1+x} - \frac{y}{1+y} \right|, \quad x, y \in [0, \infty).$$

Is it a Cauchy sequence? Why? Does it converge in the metric space $X = ([0, \infty), d)$? What does this tell us about X?

(b) Now show that $x_n \to 0$ as $n \to \infty$, with respect to the metric

$$d'(x,y) = \left| \frac{x}{1+x^2} - \frac{y}{1+y^2} \right| + \left| \frac{x}{1+2x^2} - \frac{y}{1+2y^2} \right|, \quad x, y \in [0, \infty).$$

Compare this with your argument from (a). Does it apply here? Is there any difference? A conclusion? Perhaps you need to modify your arguments in (a).

1.6. Let $X = C([-1,1])$ and for $x, y \in X$ put

$$d_1(x,y) = \int_{-1}^{1} |x(t) - y(t)|\, dt \quad \text{and} \quad d_\infty(x,y) = \max_{-1 \le t \le 1} |x(t) - y(t)|$$

Consider the functions

$$x_n(t) = \begin{cases} 1 - 2^n|t|, & \text{if } |t| \le 2^{-n}, \\ 0, & \text{otherwise} \end{cases} \qquad n = 1, 2, ...$$

and draw a few of them to get an idea of how they look like. Show the following:

(a) The sequence $(x_n)_{n=1}^{\infty}$ converges in the metric space (X, d_1).

(b) Does the sequence $(x_n)_{n=1}^{\infty}$ converge in (X, d_∞)?

1.7. (a) Use the definition of compactness (with open covers) to show that the set $K = \{0\} \cup \{2^{-n} : n = 1, 2, ...\}$ is compact.

(b) For every $\varepsilon > 0$, find finitely many balls (i.e. intervals) with radius ε so that they together cover the set $K \setminus \{0\} = \{2^{-n} : n = 1, 2, ...\}$. Which property of $K \setminus \{0\}$ have you just proved?

(c) Show that the set $K \setminus \{0\}$ in (b) is *not* compact by providing an open covering of it, which does not have a finite subcovering.

1.8. Suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d) and that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ is convergent with limit x. We want to show that $(x_n)_{n=1}^{\infty}$ is convergent with the same limit.

ChatGPT gave essentially the following answer (the text has been shortened):

Let $\varepsilon > 0$ be arbitrary. Since $(x_{n_k})_{k=1}^{\infty}$ converges to x, there exists K such that for all $k \geq K$,

$$d(x_{n_k}, x) < \frac{\varepsilon}{2}. \tag{2}$$

Because $(x_n)_{n=1}^{\infty}$ is Cauchy, there exists N such that for all $m, n \geq N$,

$$d(x_n, x_m) < \frac{\varepsilon}{2}. \tag{3}$$

Now, choose $N' := \max\{N, n_K\}$, so that:

$$n_K \geq N' \quad \textit{and} \quad n_K \geq N. \tag{4}$$

For any $n \geq N'$, we have:

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x). \tag{5}$$

Using the properties above:

$$d(x_n, x_{n_K}) < \frac{\varepsilon}{2} \quad \textit{since } n, n_K \geq N \tag{6}$$

and

$$d(x_{n_K}, x) < \frac{\varepsilon}{2} \quad \textit{since } n_K \geq n_K \textit{ and } k \geq K. \tag{7}$$

Therefore,

$$d(x_n, x) < \varepsilon. \tag{8}$$

Find at least 3 mistakes in ChatGPT's answer and explain why they are mistakes. You can mark them directly on a printout of this page. Then write a correct and shorter solution, using only (2), (3) with a smart choice of m and n_k, leading directly to (8) without the use of N'.

See next page for $*$-Problems

$*$-Problems:

1.9. Let $X = C([-1,1])$, equipped with the metrics d_1 and d_∞ defined in Exercise 1.6. Put $V = \{x \in X : |x(t)| < 1 \text{ for all } t \in [-1,1]\}$.

(a) Show that V is open in (X, d_∞).

(b) Show that the constant function $x(t) = 0$ is *not* an interior point of V with respect to d_1. What does this tell us about the set V in (X, d_1)?

1.10. (a) Assume that $(x_n)_{n=1}^\infty \subset \mathbf{c}_0$ and that $x_n \to x$ in ℓ^∞. Show that $x \in \mathbf{c}_0$. Which property of $\mathbf{c}_0$ have you just proved?

(b) Find sequences $x_n = (x_j^{(n)})_{j=1}^\infty \in \mathbf{c}_0$ such that $x_j^{(n)} \to 1$ as $n \to \infty$ for all $j = 1, 2, ...$, even though the sequence $x = (1, 1, ...)$ does *not* belong to $\mathbf{c}_0$. What does this tell you about the norm-convergence and the coordinate-wise convergence in $\mathbf{c}_0$?

Note that the superscript $^{(n)}$ just indicates which sequence it belongs to, it does *not* mean the n-th derivative! Another way of writing is in Exercise 1.12. This easily becomes an indexing nightmare but is a good exercise and hard to avoid.

1.11. Suppose that (X, d) is a metric space equipped with the discrete metric d and that sets in $\mathbf{R}$ are equipped with the standard metric.

(a) Which functions from (X, d) to $[-1, 1]$ are continuous?

(b) Which functions from $[-1, 1]$ to (X, d) are continuous?

(c) Which functions from $[-1, 1] \setminus \{0\}$ to (X, d) are continuous?

1.12. Which of the following statements defines uniform convergence (i.e. in the metric d_∞) of

$$x_n = (x_{n,1}, x_{n,2}, ...) \in l^\infty \quad \text{to} \quad x = (\bar{x}_1, \bar{x}_2, ...) \in l^\infty,$$

as $n \to \infty$? What is described by the remaining statements? For each of the statements give an example of a sequence which satisfies it.

(a) $\forall \varepsilon > 0 \quad \forall j = 1, 2, ... \quad \exists N \in \mathbf{N} \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(b) $\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \quad \forall j = 1, 2, ... \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(c) $\forall j = 1, 2, ... \quad \exists N \in \mathbf{N} \quad \forall \varepsilon > 0 \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

(d) $\exists N \in \mathbf{N} \quad \forall j = 1, 2, ... \quad \forall \varepsilon > 0 \quad \forall n \geq N \quad |x_{n,j} - \bar{x}_j| < \varepsilon$

2. Assignment 2 (Lecture 6–12 + earlier stuff)

Some of the problems have several parts and count as normal problems or as $*$-problems, depending on whether you solve the $*$-part or not.

2.1. Suppose that X is a normed space and Y is a vector (linear) subspace of X. Show that $ax + by \in \overline{Y}$ whenever $x, y \in \overline{Y}$ and $a, b \in \mathbf{R}$. Which property of the closure $\overline{Y}$ of Y (in X) have you just proved?

2.2. Suppose that X and Y are two Banach spaces, equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Recall the definitions!

(a) Prove the triangle inequality for the norm $\|z\| = \|x\|_X + \|y\|_Y$ on $X \times Y$, where $z = (x, y) \in X \times Y$.

(b) Find a mistake (and explain why it is a mistake) in the following extract from ChatGPT's attempt to prove that $X \times Y$ is complete when equipped with the above norm:

Suppose (x_n, y_n) is a Cauchy sequence in $X \times Y$. Since X and Y are Banach spaces, the sequence (x_n) is Cauchy in X and the sequence (y_n) is Cauchy in Y. Because X and Y are complete, these sequences converge: $x_n \to x$ and $y_n \to y$.

Write down a correct proof that $X \times Y$ is complete by completing the statements in the boxes (you can write this on a separate paper if you want):

Let $z_n = (x_n, y_n)$ be a Cauchy sequence in $X \times Y$.
$\Updownarrow$ (definition of Cauchy sequence)
$\forall \varepsilon > 0$
$\Downarrow$ (definition of norm on $X \times Y$)
$\forall \varepsilon > 0$
$\Downarrow$ (motivation:)
...
$\Downarrow$ (X, Y Banach spaces)
$\Downarrow$ (motivation:)
$z :=$ ... and $\|z_n - z\| =$
$\Downarrow$ (motivation:)
$z_n \to z$ in $X \times Y$.

2.3. Suppose that the sequences

$$x_n = (x_1^{(n)}, x_2^{(n)}, \ldots) \to x \text{ in } \ell^p \quad \text{and} \quad y_n = (y_1^{(n)}, y_2^{(n)}, \ldots) \to y \text{ in } \ell^{p'},$$

where p' is the conjugate exponent to $p \in [1, \infty]$. Show that $x_n y_n \to xy$ in ℓ^1. Here, the products between sequences are defined coordinate-wise, i.e.

$$x_n y_n = (x_1^{(n)} y_1^{(n)}, x_2^{(n)} y_2^{(n)}, \ldots).$$

Remember that the superscript $^{(n)}$ just indicates which sequence it belongs to, *not* the n-th derivative!

2.4. Calculate the limit $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{1}{(x^n + x^{4n})^{1/2n}}\, dx.$

2.5. Let $f_n(x) = n/(1 + x^n)$ for $x > 0$. Calculate the limits

$$f(x) := \lim_{n\to\infty} f_n(x) \qquad \text{and} \qquad \lim_{n\to\infty} \int_1^\infty f_n(x)\, dx.$$

Is the latter limit equal to $\displaystyle\int_1^\infty f(x)\, dx$?

2.6. Let $\|\cdot\|_p$ denote the norm in ℓ^p, $1 \le p \le \infty$.

(a) Show that $\|x\|_\infty \le \|x\|_p$ if $1 \le p < \infty$.

(b) Show that $\|x\|_q \le \|x\|_p$ if $1 \le p \le q < \infty$.

(c) What do (a) and (b) tell you about the inclusion between ℓ^p and ℓ^q?

2.7. Find a continuous function f such that $\int_{\mathbf{R}} |f(x)|\, dx < \infty$ and $f(x) \not\to 0$ as $x \to \infty$.

2.8. Let $(r_n)_{n=1}^\infty$ be an enumeration of all the rational numbers in $(0, 1)$ and put

$$f(x) = \sum_{r_n < x} \frac{2^{-n}}{1 - r_n}, \quad x \in [0, 1].$$

Show that f is measurable and calculate the integral $\int_0^1 f(x)\, dx$.

2.9. Put $f_n(t) = \sin nt$, $t \in (-\pi, \pi)$, $n = 1, 2, \ldots$, and $S = \{f_n : n = 1, 2, \ldots\}$.

(a) Show that S is a bounded subset of $L^2(-\pi, \pi)$.

(b) Show that S is a closed subset of $L^2(-\pi, \pi)$.

(c) (For a "$*$") Is S compact?

$*$-Problems:

2.10. Suppose that $f \in L^1(\mathbf{R})$. Show that the integral

$$F(x) = \int_0^x f(t)\, dt, \quad x \in \mathbf{R},$$

is continuous on $\mathbf{R}$, i.e. that $F(x_n) \to F(x)$ when $x_n \to x$.

2.11. Find a function $f \notin L^\infty(0, 1)$ such that $f \in L^p(0, 1)$ for all $1 \le p < \infty$.

2.12. Suppose that $f \in L^\infty(0, 1)$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

3. Assignment 3 (Lecture 13–17 + earlier stuff)

Some of the problems have several parts and count as normal problems or as $*$-problems, depending on whether you solve the $*$-part or not.

3.1. For $x \in L^2(-1,1)$ find constants a and b, which minimize the expression

$$\int_{-1}^{1} |x(t) - (a+bt)|^2 \, dt,$$

that is, find the best approximation of x in $L^2(-1,1)$ by a linear function. What are a and b for $x(t) = t^n$, $n = 2, 3, ...$?

3.2. Suppose that X is an inner-product space.

(a) If $M_1 \subset M_2 \subset X$, which inclusions hold for $M_1^\perp$ and $M_2^\perp$, and for $(M_1^\perp)^\perp$ and $(M_2^\perp)^\perp$?

(b) Let $(x_n)_{n=1}^\infty$ be a sequence in X such that $x_n \to x$. Show that if $x_n \perp y$ for some $y \in X$ and all $n = 1, 2, ...$, then $x \perp y$. Which property of

$$M^\perp = \{x \in X : x \perp y \text{ for all } y \in M\}$$

follows from this for an arbitrary set $M \subset X$?

3.3. Let $S = \{x \in L^2(-1,1) : \int_0^1 t x(t) \, dt = -1\}$. Show that S contains a unique element with minimal L^2-norm and find this element. Justify the uniqueness.

3.4. (a) Let α and β be two fixed numbers. Show that

$$g(x) = \alpha x(0) + \beta \int_0^1 x(t^2) \, dt, \quad x \in C([0,1])$$

is a bounded linear *functional on* the space $X := (C([0,1]), \|\cdot\|_\infty)$, i.e. from X to $\mathbf{R}$ (or $\mathbf{C}$).

(b) Show that

$$T : x \mapsto t^2 \int_a^t x(s) \, ds, \quad x \in C([a,b])$$

is a bounded linear *operator on* the space $X := (C([a,b]), \|\cdot\|_\infty)$, i.e. from X to X. Give a concrete (not necessarily the best, but correct!) estimate of the norm $\|T\|$ in terms of a and b. (Do not calculate $\|T\|$ exactly, just estimate.)

3.5. Let $T : X \to Y$ be an invertible bounded linear operator between normed spaces X and Y. Show that $\|T^{-1}\| \geq \|T\|^{-1}$. Calculate $\|T\|$ and $\|T^{-1}\|$ for $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1, 2x_2)$. What does this tell us about the inequality $\|T^{-1}\| \geq \|T\|^{-1}$?

3.6. Let $1 < p < \infty$ and let L denote the left-shift operator on ℓ^p, i.e.

$$L(x_1, x_2, ...) = (x_2, x_3, ...).$$

(a) Is L injective, i.e. does $Lx = Ly$ imply $x = y$?

(b) Is L surjective, i.e. $L(\ell^p) = \ell^p$?

(c) Calculate $\|L^n\|$, $\lim_{n\to\infty} \|L^n\|$ and $\lim_{n\to\infty} \|L^n x\|_p$, where $x \in \ell^p$ and L^n denotes the composition of L with itself $n-1$ times.

(d) What does (c) say about the pointwise and the norm convergence of the operators L^n, as $n \to \infty$?

3.7. On $X = \{x \in C^1([0,2]) : x(0) = 0\}$, consider the linear functional $f(x) = x(2)$.

(a) Show that f is bounded with respect to the norm $\|x\|_* := \left(\int_0^2 |x'|^2\, dt\right)^{1/2}$.

(b) Write f as $f(x) = (x, z)_*$ for a suitable function $z \in X$, where $(\cdot, \cdot)_*$ is the inner product associated with the above norm $\|\cdot\|_*$.

(c) Calculate the norm of f with respect to $\|\cdot\|_*$ and with respect to $\|\cdot\|_\infty$.

(d) Does the norm $\|\cdot\|_\infty$ come from an inner product? Why?

3.8. Let $Z = \{x \in L^2(-1,1) : \int_{-1}^1 x(t)\, dt = 0\}$.

(a) Show that Z is a closed linear subspace of $L^2(-1,1)$.

(b) (For a "$*$") Show that $Y := Z^\perp$ consists exactly of all constant functions. Given $x \in L^2(-1,1)$, determine the orthogonal decomposition $x = x_Y + x_Z$, where $x_Y \in Y$ and $x_Z \in Z$.

$*$-Problems:

3.9. Calculate the norm $\|f\|$ of the linear functional f on $(C([-1,1]), \|\cdot\|_\infty)$ defined by

$$f(x) = \int_0^1 x(t)\, dt - \int_{-1}^0 x(t)\, dt, \quad x \in C([-1,1]).$$

Is there $x \in C([-1,1])$ such that $\|x\|_\infty = 1$ and $f(x) = \|f\|$? If not, justify why!

3.10. Show that if f is a bounded linear functional on $\mathbf{c}_0$ then there is $y \in \ell^1$ such that

$$f(x) = \sum_{j=1}^\infty x_j y_j \quad \text{for all } x \in \mathbf{c}_0.$$

3.11. Suppose that H is a Hilbert space and let $(e_n)_{n=1}^\infty$ be an infinite ON-sequence in H. Put $Y = \text{span}\{e_1, e_2, ...\}$, i.e. Y consists of all *finite* linear combinations of the e_n's.

(a) Show that if $x = \sum_{n=1}^\infty (x, e_n) e_n$ then $x \in \overline{Y}$.

(b) Show that if $x \in \overline{Y}$ then $x = \sum_{n=1}^\infty (x, e_n) e_n$.

Hints

1.1. Remember that every set is the union of all of its points. In (b), remember that "if and only if" is an equivalence, so you need to prove two implications. Which ones? One of them is similar to part (a). Be careful about the choice of the balls in the union. What do we know about unions of open sets?

1.2. Find disjoint sets $A, B \subset \mathbf{R}$ such that $\overline{A}$ and $\overline{B}$ are not disjoint.

1.3. Assume that the conclusion is not true and deduce that x is then not an accumulation point of A. Do not ignore the condition $y \neq x$ in the definition of accumulation points!

1.4. A picture may help. For an arbitrary $x \in X$, what can you say about $d(x, x_j)$ for $j = 1, \dots, N$?

1.5. (a) Assume that your sequence converges in the metric d to some $x \in [0, \infty)$. What can you then say about x? Does it exist? Use only points from the metric space $[0, \infty)$ and convergence with respect to the metric d, i.e. *not* the usual Euclidean metric on $\mathbf{R}$. Note that $n \to \infty$ (n as index) means something different than $x_n \to \infty$ (which does not make sense since ∞ is not a point in X)!

1.6. (a) A good candidate for the limit is the constant function $x(t) = 0$. Why? For a simple integration, recall that integral corresponds to an area.

(b) Which other useful convergence follows from the d_∞-convergence?

1.7. In (a), is there a set G_{λ_0} in the open cover of K such that $0 \in G_{\lambda_0}$? Is it useful? Pictures may help.

1.9. (a) For example, you can show that every $x \in V$ is an interior point of V with respect to d_∞. You can even find a concrete $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V$.

(b) What is $d_1(0, y)$ for a function $y \in (X, d_1)$? Can you make this small while $y \notin V$?

1.10. Look at the proof in the lectures showing that uniform convergence preserves continuity. Do not mix the elements $x_j^{(n)}$ in each sequence x_n with the sequences themselves as members in a converging sequence $x_n \to x$. Keep track of the indices. Remember that the speed of convergence in $x_j^{(n)} \to 0$, as $j \to \infty$, is different for different n.

1.11. In (a) use some of the equivalent characterizations of continuity that we had in the lectures, together with some of the following questions: Which sets in (X, d) are open? If $x_n \to x$ in (X, d), what can we say about x_n for sufficiently large n? In (b) and (c), you might want to look at what happens near the point

$$x_0 := \inf\{x : f(x) \neq f(-1)\}.$$

What is the difference between $[-1, 1]$ and $[-1, 1] \setminus \{0\}$ in this situation?

2.1. Recall the definition of vector spaces and that $x \in \overline{Y}$ if and only if there exist $x_n \in Y$ such that $\|x_n - x\| \to 0$. If you want, you can structure your proof using similar boxes and implications as in Exercise 2.2.

2.3. Somewhere you will need to use Hölder's inequality (twice, actually). It is useful to write it as $\|xy\|_1 \leq \|x\|_p \|y\|_{p'}$. What does it mean that $x_n y_n \to xy$ in $\|\cdot\|_1$? Inserting $\pm x_n y$ and the triangle inequality also help. Recall that converging sequences are bounded.

2.4. We have convergence theorems for the Lebesgue integral. Split the integral at $x = 1$ into two integrals. In each of them, which nice *integrable* function dominates the integrand?

2.5. We have convergence theorems for the Lebesgue integral. Begin by changing variables and remember that you are taking a limit as $n \to \infty$. So, how important are the first $n = 1, 2, 3$ for the limit? For which n is $f_n \in L^1(1, \infty)$?

2.6. In (a), compare $|x_j|$ and $\|x\|_p$. If you use $\max_j |x_j|^p$, explain why it exists! Using sup is easier and safer. In (b), write $|x_j|^q = |x_j|^{q-p} |x_j|^p$ and compare with $\|x\|_p$ and $\|x\|_q$. In (c) explain why $x \in$ "smaller space" implies $x \in$ "bigger space". You have done most of the work already.

2.7. Recall that $f(x) \not\to 0$ means that either $\lim_{x\to\infty} f(x) \to c \neq 0$ or that

$$\limsup_{x\to\infty} f(x) > \liminf_{x\to\infty} f(x).$$

Exclude one of the possibilities and note that f will not be monotone.

2.8. Write $f(x)$ as a limit of partial sums with $\chi_{(r_n,1)}$. How can we make new measurable functions out of old ones? *Warning*: All rational numbers in $[0,1]$ *cannot* be ordered in a *monotone* sequence, so do *not* use that.

2.9. What is $\|f_n - f_m\|_2$? Identify all accumulation points for S. Or, think about which are the converging sequences in S? Use some of the characterizations of closed and compact sets that we had in the lectures.

2.10. Write $F(x_n) - F(x)$ as an integral and rewrite it as an integral over $\mathbf{R}$. We have convergence theorems for the Lebesgue integral. How do you choose f_n? *Warning*: The convergence $x_n \to x$ need not be monotone.

2.11. For which $\alpha < 0$ is the power $x^\alpha \in L^p(0,1)$? Can you find $0 \leq f \notin L^\infty(0,1)$ which is majorized at 0 by all such powers?

2.12. One inequality is easy. For the other inequality and a fixed $\varepsilon > 0$ consider the set $A := \{x : |f| \geq (1-\varepsilon)\|f\|_\infty\}$. Estimate $\|f\|_p$ from below using A.

3.1. Use orthogonal projections and do not forget to normalize your ON-basis properly.

3.2. In (a), use the definition of $M^\perp$. Remember that M_1 and M_2 need not be linear subspaces. In (b), try $x_n \in M^\perp$ and $y \in M$.

3.3. Use inner product on $L^2(-1,1)$ and the Cauchy–Schwarz inequality with a suitable function $y \in L^2(-1,1)$ to get a lower bound for $\|x\|_2$ for any $x \in S$. When do we have equality in the Cauchy–Schwarz inequality?

3.4. Recall the definition of bounded linear operators and functionals. Do not abuse absolute values. Warning: a and b can be negative or zero!

3.5. Use e.g. the supremum formula for calculating $\|T\|$ and $\|T^{-1}\|$. You can also use that $T^{-1}T = I$ but if you want to use formula (7) on p. 98 in Kreyszig, then you need to prove it first (which is basically to use the supremum formula).

3.6. To calculate the norms, you need estimates both from above and from below. Check carefully the definition, so that you are calculating the correct expressions and comparing correct convergences.

3.7. Rewrite f as an integral. Which equality must hold for inner product norms?

3.8. In (a), you can use the inner product. Consider e.g. $x_n \in Z$ such that $x_n \to x$ in $L^2(-1,1)$ and show that $x \in Z$. Justify carefully convergence of the integrals. In (b), write Z as an orthogonal complement.

3.9. First, estimate $|f(x)|$ using $\|x\|_\infty$. Then find $x_n \in C([-1,1])$ with $\|x_n\|_\infty = 1$ such that $f(x_n) \to \|f\|$. If $\|x\|_\infty = 1$, which estimates must hold for the integrals in the definition of $f(x)$?

3.10. See the proof from the lecture showing that the dual of ℓ^p is $\ell^{p'}$.

3.11. (a) Consider the partial sums in the above infinite series.

(b) Consider $z = x - \sum_{n=1}^\infty (x, e_n)e_n$ and show that $z \perp Y$. Conclusion?