Bengt Ove Turesson

Functional Analysis

Exercises

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Linköpings universitet

Contents

1. Metric Spaces

1.1. Suppose that (X,d) is a metric space and put

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \quad \text{for } x,y \in X.$$

Show that d_1 is a metric on X.

1.2. Show that a non-empty subset of a metric space is open if and only if it is a union of open balls.

1.3. Show that the only subsets of $\mathbf{R}$, equipped with the standard metric, which are both open and closed, are $\mathbf{R}$ and $\varnothing$.

1.4. Suppose that A and B are two subsets of a metric space. Show that

$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Show by an example that the inclusion may be strict.

1.5. Give an example of a metric space (X,d) where the closure of the open ball $B_r(x)$ not necessarily coincides with the closed ball $\overline{B}_r(x)$.

1.6. Suppose that (X,d) is a metric space and that A is a nonempty subset of A. Show that if x is an accumulation point of A, then any neighbourhood of x contains infinitely many points of A.

1.7. Suppose that (X,d) is a discrete metric space and that $\mathbf{R}$ is equipped with the standard metric. Which functions from X to $\mathbf{R}$ are continuous? Which functions from $\mathbf{R}$ to X are continuous?

1.8. Suppose that (X,d_X) and (Y,d_Y) are two metric spaces and let $T : X \to Y$ be a mapping from X to Y. Show that T is continuous if and only if the inverse image $T^{-1}(F)$ of any closed subset F of Y is a closed subset of X.

1.9. Show that $\mathbf{c}_0$ is separable. Notice in contrast that ℓ^∞ is not separable.

1.10. Give an example of a sequence $x \in \mathbf{c}_0$ such that x does not belong to ℓ^p for any number $1 \le p < \infty$.

1.11. Suppose that $x_n \to x$ in ℓ^p and $y_n \to y$ in $\ell^{p'}$. Show that $x_n y_n \to xy$ in ℓ^1. Here, the products between sequences are defined coordinate-wise.

1.12. (a) Show that $\mathbf{c}_0$ is a closed subset of ℓ^∞.

(b) Suppose that $x_n = (x_j^{(n)})_{j=1}^\infty$ is a sequence in $\mathbf{c}_0$, which has the property that $x_j^{(n)} \to x_j$ as $n \to \infty$ for $j = 1, 2, \ldots$. Does it follow that the sequence $x = (x_j)_{j=1}^\infty$ belongs to $\mathbf{c}_0$?

1.13. Suppose that (x_n) is a Cauchy sequence in a metric space and that a subsequence of (x_n) is convergent with limit x. Show that (x_n) is convergent with the same limit.

1.14. Show that any Cauchy sequence in a metric space is bounded.

1.15. Using the fact that $\mathbf{R}$ is complete, show that $\mathbf{C}$ is complete.

1.16. Show that $\mathbf{c}_{00}$ is not complete.

1.17. Show that $\mathbf{R}$ is not complete equipped with the metric

$$d(x,y) = |\arctan x - \arctan y|, \quad x, y \in \mathbf{R}.$$

1.18. Let $X = C([0,1])$ and put

$$d_1(x,y) = \int_0^1 |x(t) - y(t)|\,dt \quad \text{and} \quad d_\infty(x,y) = \max_{0 \le t \le 1} |x(t) - y(t)|$$

for $x, y \in X$. Also put $x_n(t) = nte^{-nt}$, $0 \le t \le 1$ for $n = 0, 1, \ldots$.

(a) Is it true that (x_n) is convergent in the metric space (X, d_1)?
(b) Is it true that (x_n) is convergent in the metric space (X, d_∞)?

1.19. Let $X = C([0,1])$ and put $A = \{x \in X : |x(t)| < 1 \text{ for } 0 \le t \le 1\}$.

(a) Is A open in (X, d_1)?
(b) Is A open in (X, d_∞)?

Here, d_1 and d_∞ are the metrics defined in Exercise 1.18.

1.20. Show that any discrete metric space (X, d), such that X has infinitely many elements, is not compact.

2. Normed Spaces

2.1. For which values of α is

(a) $\phi(x) = |x|^\alpha$, $x \in \mathbf{R}$, a norm on $\mathbf{R}$?
(b) $d(x,y) = |x - y|^\alpha$, $x, y \in \mathbf{R}$, a metric on $\mathbf{R}$?

2.2. Show that any ball in a normed space is convex.

2.3. Show that the discrete metric on a vector space $X \neq \{0\}$ cannot be obtained from a norm.

2.4. Suppose that X and Y are two Banach spaces.

(a) Verify that $\|(x,y)\| = \|x\| + \|y\|,\ (x,y) \in X \times Y$, is a norm on $X \times Y$.

(b) Show that $X \times Y$ is a Banach space equipped with this norm.

2.5. Let $a_j,\ j = 1,2,\ldots$, be non-negative real numbers and suppose that $0 < \alpha \le 1$. Show that

(a) $(1 + a_1)^\alpha \le 1 + a_1^\alpha$;

(b) $(a_1 + a_2)^\alpha \le a_1^\alpha + a_2^\alpha$;

(c) $\left(\sum_{j=1}^n a_j\right)^\alpha \le \sum_{j=1}^n a_j^\alpha$ for $n = 1,2,\ldots$;

(d) $\left(\sum_{j=1}^\infty a_j\right)^\alpha \le \sum_{j=1}^\infty a_j^\alpha$ if the series $\sum_{j=1}^\infty a_j^\alpha$ is convergent.

2.6. Let $\|\cdot\|_p$ denote the norm in $\ell^p,\ 1 \le p \le \infty$.

(a) Show that $\|x\|_q \le \|x\|_p$ if $1 \le p \le q < \infty$.

(b) Show that $\|x\|_\infty \le \|x\|_p$ if $1 \le p < \infty$ and moreover that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$.

(c) Conclude that $\ell^p \subset \ell^q$ for $1 \le p \le q \le \infty$.

2.7. Suppose that X is a normed spaced and Y is a subspace of X. Show that the closure $\overline{Y}$ of Y is a subspace of X.

2.8. Show that $e_n = (\delta_{jn})_{j=1}^\infty,\ n = 1,2,\ldots$, is a Schauder basis for ℓ^p for $1 \le p < \infty$. Here, δ_{jn} is the *Kronecker delta* defined by $\delta_{jn} = 1$ if $j = n$ and $\delta_{jn} = 0$ otherwise.

2.9. Show that if a normed space has a Schauder basis, then the space is separable.

3. Theory of Integration

3.1. Show that the set $K = \{0\} \cup \{n^{-1} : n = 1,2,\ldots\}$ is compact.

3.2. Let $(r_n)_{n=1}^\infty$ be an enumeration of the rational numbers in $[0,1]$ and put

$$I_n = \left(r_n - \frac{1}{\pi^2 n^2}, r_n + \frac{1}{\pi^2 n^2}\right) \quad \text{for } n = 1,2,\ldots.$$

Is $(I_n)_{n=1}^\infty$ an open covering of $[0,1]$?

3.3. Construct a sequence $(\phi_n)_{n=1}^\infty$ of nonnegative step functions on $[0,1]$ such that the numerical sequence $(\phi_n(x))_{n=1}^\infty$ is not convergent for any $x \in [0,1]$, while

$$\int_0^1 \phi_n(x)\,dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

3.4. Suppose that $f \in L^1(\mathbf{R}) \cap C(\mathbf{R})$. Is it true that $f(x) \to 0$ as $x \to \pm\infty$?

3.5. The so-called *sinc function* is defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases} .$$

Show that $f \notin L^1(\mathbf{R})$. Notice, however, that f is generalized Riemann integrable.

3.6. Suppose that $f \in L^1(\mathbf{R})$. Show that the integral

$$F(x) = \int_0^x f(t)\,dt, \quad x \in \mathbf{R},$$

is continuous on $\mathbf{R}$.

3.7. Suppose that $f \in \text{Ł}^1(-\pi, \pi)$. Show that

$$\int_{-\pi}^{\pi} f(t) e^{int}\,dt \longrightarrow 0 \quad \text{as } n \to \pm\infty.$$

3.8. Calculate the limit

$$\lim_{n\to\infty} \int_0^1 \frac{n^{3/2} x}{1 + n^2 x^2}\,dx.$$

3.9. Calculate the limit

$$\lim_{n\to\infty} \int_0^1 \frac{n x^2}{(1 + x^2)^n}\,dx.$$

3.10. Calculate the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{1}{(x^n + x^{4n})^{1/2n}}\,dx.$$

3.11. Calculate the limit

$$\lim_{n\to\infty} \int_1^\infty \frac{n}{1 + x^n}\,dx.$$

3.12. Let $(r_n)_{n=1}^\infty$ be an enumeration of the rational numbers in $[0, 1]$.

(a) Show that the series

$$g(x) = \sum_{n=1}^\infty \frac{1}{n^2 \sqrt{|x - r_n|}}, \quad x \in [0, 1],$$

is convergent for a.e. $x \in [0, 1]$.

(b) Show that g is unbounded on every subinterval of $[0, 1]$.

(c) Show that g does not belong to $L^2(0, 1)$.

3.13. Calculate the limit

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - \int_0^\infty \frac{1}{1+x^2+x^n}\,dx\right).$$

3.14. Let $(r_n)_{n=1}^\infty$ be an enumeration of the rational numbers in $(0,1)$ and put

$$f(x) = \sum_{r_n<x} 2^{-n}, \quad x \in [0,1].$$

Show that $f \in L^1(0,1)$ and calculate the integral $\int_0^1 f(x)\,dx$.

3.15. Find an unbounded function f on $(0,1)$ such that $f \in L^p(0,1)$ for $1 \le p < \infty$.

3.16. Suppose that $f \in L^\infty(0,1)$. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

3.17. Put $f_n(x) = \sin nx,\ x \in (-pi, \pi),\ n = 1, 2, \dots,$ and $M = \{f_n : n = 1, 2, \dots\}$.

(a) Show that M is a closed and bounded subset of $L^2(-\pi, \pi)$.
(b) Is M compact?

4. Inner-product Spaces

4.1. Suppose that X is an inner-product space. Show that the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for all vectors $x, y \in X$.

4.2. Can the norm $\|x\| = |x_1| + |x_2|$ on $\mathbf{R}^2$ be obtained from an inner-product?

4.3. Suppose that X is an inner-product space and (x_n) is a sequence in X. Show that if $x_n \to x$ and $x_n \perp y$, then $x \perp y$.

4.4. Determine the orthogonal complement in ℓ^2 of the following subspaces of ℓ^2:

(a) $Y = \{x \in \ell^2 : x_2 = x_4 = \dots = 0\}$
(b) $Y = \mathrm{span}\{e_1, \dots, e_k\}$, where $k \ge 1$ and $(e_n)_{n=1}^\infty$ is the standard basis for ℓ^2.

4.5. Suppose that H is a Hilbert space and M is a non-empty subset of H. Show that the set $M^{\perp\perp}$ is the smallest closed subspace of H that contains M. More precisely, show that $M^{\perp\perp}$ is a closed subspace of H and moreover that $M^{\perp\perp} \subset Y$ for any closed subspace of H such that $M \subset Y$.

4.6. Suppose that H is a Hilbert space and let $(e_n)_{n=1}^\infty$ be an infinite orthonormal sequence in H. Put $Y = \mathrm{span}\{e_1, e_2, \dots\}$. Show that $x \in \overline{Y}$ if and only if

$$x = \sum_{n=1}^\infty (x, e_n) e_n.$$

4.7. Suppose that H is a Hilbert space and let M be a countable dense subset of H. Show that the Gram–Schmidt process applied to M produces an orthonormal basis for H.

4.8. Suppose that H is a separable Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. Show that

$$(x, y) = \sum_{n=1}^{\infty} (x, e_n)\overline{(y, e_n)}$$

for all vectors $x, y \in H$.

4.9. Let X be an inner-product space and $y \in X$. Show that $f(x) = (x, y),\ x \in X$, is a bounded linear functional on X with norm $\|y\|$.

4.10. Show that

$$\left|\sum_{j=1}^{\infty} 2^{-j/2} x_j\right| \leq \|x\|_2$$

for any vector $x \in \ell^2$. For which vectors x does equality occur in this inequality?

4.11. The unit ball in a normed space X is said to be *strictly convex* if

$$\|tx + (1-t)y\| < 1 \quad \text{whenever } 0 < t < 1$$

for all distinct vectors x and y on the unit sphere of X. It can be shown that the unit ball in $L^p(a, b)$ is strictly convex for $1 < p < \infty$.

(a) Show that unit ball in a inner-product space X is strictly convex. In particular, the unit ball in $L^2(a, b)$ is strictly convex.

(b) Show that unit balls in $L^1(0, 1)$ and $L^\infty(0, 1)$ are not strictly convex.

4.12. Let $M = \{x \in L^1(0, 1) : \int_0^1 x(t)\, dt = 1\}$. Show that

(a) M is a closed, convex subset of $L^1(0, 1)$;

(b) M contains infinitely many elements with minimal norm, that is, there exist infinitely many $x \in M$ with $\|x\|_1 = \inf_{y \in M} \|y\|_1$.

4.13. Let $M = \{x \in L^2(0, 1) : \int_0^1 x(t)\, dt = 1\}$. Show that

(a) M is a closed, convex subset of $L^2(0, 1)$;

(b) M contains a unique element x with minimal norm and determine this element.

4.14. (a) Show that $\mathbf{c}_{00}$ is not closed in ℓ^2.

(b) Determine $\mathbf{c}_{00}^{\perp}$ and $\mathbf{c}_{00}^{\perp\perp}$.

4.15. Let $M = \{x \in L^2(0, 1) : \int_0^1 x(t)\, dt = 0\}$.

(a) Show that M is a closed subspace of $L^2(0, 1)$.

(b) Determine $M^{\perp}$ and $M^{\perp\perp}$.

(c) Given $x \in L^2(0, 1)$, determine the orthogonal decomposition $x = y + z$, where $y \in M$ and $z \in M^{\perp}$.

5. Linear Operators

5.1. Suppose that X and Y are two vector spaces and that $T : X \to Y$ is a linear operator.

(a) Show that the null-space $N(T)$ of T is a subspace of X.

(b) Show that the range $R(T)$ is a subspace of Y.

5.2. Suppose that X and Y are two vector spaces and that $T : X \to Y$ is a linear operator.

(a) Show that $T(V)$ is a subspace of Y for any subspace V of X.

(b) Show that $T^{-1}(W)$ is a subspace of X for any subspace W of Y.

5.3. Show that the composition of two linear operators is linear.

5.4. Show that the inverse of a linear operator is linear.

5.5. (a) Show that

$$f(x) = \int_a^b x(t)y(t)\,dt, \quad x \in C([a,b]),$$

where $y \in C([a,b])$ is a fixed function, is a bounded linear functional on $C([a,b])$.

(b) Show that

$$g(x) = \alpha x(a) + \beta x(b), \quad x \in C([a,b]),$$

where α and β are two fixed numbers is a bounded linear functional on the space $C([a,b])$.

5.6. Calculate the norm of the linear functional T on $C([-1,1])$ defined by

$$Tx = \int_0^1 x(t)\,dt - \int_{-1}^0 x(t)\,dt, \quad x \in C([-1,1]).$$

5.7. Let S and T denote bounded linear operators on a normed space X.

(a) Show that $\|ST\| \leq \|S\|\|T\|$. Give an example where this inequality is strict.

(b) Show that $\|T^n\| \leq \|T\|^n$ for $n = 1, 2, \ldots$. Here, T^n denotes composition of T with itself $n-1$ times.

5.8. Suppose that T is a bounded linear operator from a normed space X onto a normed space Y and furthermore that there exists a constant $C > 0$ such that

$$\|Tx\| \geq C\|x\| \quad \text{for every vector } x \in X.$$

Show that T is invertible and T^{-1} is bounded.

5.9. Define $T : C([0,1]) \to C([0,1])$ by

$$Tx(t) = \int_0^t x(s)\,ds, \quad 0 \leq t \leq 1,$$

where $x \in C([0,1])$.

(a) Show that T is bounded and calculate the norm of T.

(b) Show that T is injective.

(c) Determine the range $R(T)$ of T. Is $R(T)$ closed in $C([0,1])$?

(d) Determine $T^{-1} : R(T) \to C([0,1])$. Is T^{-1} bounded?

5.10. Let T denote the left-shift operator on ℓ^2.

(a) Is T injective?

(b) Is T surjective?

(c) Calculate the limits $\lim_{n\to\infty} \|T^n x\|_2$, where $x \in \ell^2$, and $\lim_{n\to\infty} \|T^n\|$.

5.11. The operator $T : \ell^\infty \to \ell^\infty$ is defined by $Tx = (j^{-1}x_j)_{j=1}^\infty,\ x \in \ell^\infty$.

(a) Show that T is bounded and calculate the norm of T.

(b) Show that T is injective.

(c) Determine the range $R(T)$ of T. Is $R(T)$ is closed in ℓ^∞?

(d) Determine $T^{-1} : R(T) \to \ell^\infty$. Is T^{-1} bounded?

5.12. Define $T : C^1([0,1]) \to \mathbf{C}$ by $Tx = x'(0),\ x \in C^1([0,1])$.

(a) Show that T is bounded and determine the norm of T.

(b) Does there exists a nonzero function $x \in C^1([0,1])$ such that $|Tx| = \|T\|\|x\|$?

5.13. Suppose that X and Y are normed spaces such that $\dim(X) = \infty$ and $Y \neq \{0\}$. Show that there exists at least one unbounded linear operator $T : X \to Y$.

6. Dual Spaces

6.1. Suppose that t_0 is a fixed number such that $0 \leq t_0 \leq 1$ and put

$$f(x) = x(t_0), \quad x \in C([0,1]).$$

Show that $f \in C([0,1])'$ and determine the norm of f.

6.2. Show that the dual space of $\mathbf{c}_0$ is ℓ^1.

7. The Hahn–Banach Theorem, the Banach–Steinhaus Theorem, the Open Mapping Theorem, the Closed Graph Theorem

7.1. Consider the subspace $Y = \{x \in \mathbf{R}^3 : x_3 = 0\}$ of $\mathbf{R}^3$ and define the linear functional $f : Y \to \mathbf{R}$ by $f(x) = a \cdot x,\ x \in Y$, where $a = (a_1, a_2, 0)^t \in \mathbf{R}^3$ is a fixed vector. Determine all norm-preserving linear extensions F of f to $\mathbf{R}^3$.

7.2. Let Y denote the subspace of $C([0,1])$, consisting of constant functions. Give an example of a bounded, linear functional on Y, which has infinitely many norm-preserving linear extensions to $C([0,1])$.

7.3. Let f be a bounded, linear functional on a closed subspace $Y \neq \{0\}$ of a Hilbert space H. Show that f has a unique norm-preserving linear extension F to H.

7.4. Let ρ be a seminorm on a vector space X. Show that there for any given vector $x_0 \in X$ exists a linear functional f on X such that $f(x_0) = \rho(x_0)$ and $|f(x)| \leq \rho(x)$ for every vector $x \in X$.

7.5. Suppose that X is a normed space. Show that if $f(x) = f(y)$ for every $f \in X'$, then $x = y$.

7.6. Show that any closed subspace of a reflexive Banach space is reflexive.

7.7. Show that a Banach space is reflexive if and only if its dual space is reflexive.

7.8. Suppose that X is a normed space and let M be any subset of X. Show that a vector $x \in X$ belongs to $\overline{\operatorname{span}(M)}$ if and only if $f(x) = 0$ for every $f \in X'$ such that $f = 0$ on M.

7.9. Suppose that $x \in \ell^\infty$. Show that if the series $\sum_{j=1}^\infty x_j y_j$ is convergent for every sequence $y \in \ell^2$, then actually $x \in \ell^2$.

7.10. Suppose that $x \in \ell^\infty$. Show that if the series $\sum_{j=1}^\infty x_j y_j$ is convergent for every sequence $y \in \mathbf{c}_0$, then actually $x \in \ell^1$.

7.11. Suppose that X is a Banach space and (x_n) is a sequence in X such that $(f(x_n))$ is bounded for every $f \in X'$. Show that (x_n) is bounded.

7.12. Investigate if the following operators are open.

(a) $T : \mathbf{R}^2 \to \mathbf{R}$ defined by $Tx = x_1$ for $x \in \mathbf{R}^2$;

(b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $Tx = (x_1, 0)$ for $x \in \mathbf{R}^2$.

7.13. Suppose that X and Y are two Banach spaces and $T \in B(X, Y)$ is injective. Consider the operator $T^{-1} : R(T) \to X$. Show that T^{-1} is bounded if and only if $R(T)$ is a closed subspace of Y.

7.14. Suppose that X and Y are two normed spaces and $T : X \to Y$ is a closed operator.

(a) Show that $T(K)$ is closed in Y for every compact subset K of X.

(b) Show that $T^{-1}(K)$ is closed in X for every compact subset K of Y.

7.15. Show that if $T : X \to Y$ is a closed operator, where X and Y are two normed spaces and Y is compact, then T is bounded.

7.16. Suppose that X and Y are two normed spaces, where X is compact. Show that if $T : X \to Y$ is a bijective, closed operator, then T^{-1} is bounded.

7.17. Show that if $T : X \to Y$ is a closed operator, where X and Y are two normed spaces, then the null space $N(T)$ of T is closed.

7.18. Suppose that X and Y are two normed spaces, $S : X \to Y$ is a closed operator, and $T : X \to Y$ is a bounded operator. Show that $S + T$ is closed.

8. Weak and Weak* Convergence

8.1. Show that if $x_n \to x$ weakly in $C([a, b])$, then $x_n \to x$ pointwise.

8.2. Suppose that $x_n \to x$ weakly and $y_n \to y$ weakly in a normed space X. Show that $\alpha x_n + \beta y_n \to \alpha x + \beta y$ weakly in X for all numbers α and β.

8.3. Put $x_n = (0, \dots, 0, 1, \dots, 1, 0, \dots),\ n = 1, 2, \dots$, where the ones are placed in entry n to $2n$.

(a) Show that (x_n) converges weakly to 0 in $\mathbf{c}_0$.

(b) Is the sequence convergent in $\mathbf{c}_0$?

8.4. The sequence $(x_n) \subset L^2(\mathbf{R})$ is defined by

$$x_n(t) = \begin{cases} \sqrt{n} & \text{for } -\dfrac{1}{2n} < t < \dfrac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

for $n = 1, 2, \dots$.

(a) Show that (x_n) converges weakly to 0 in $L^2(\mathbf{R})$.

(b) Is the sequence convergent in $L^2(\mathbf{R})$?

8.5. Suppose that $x_n \to x$ weakly in a normed space X. Show that

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

8.6. Suppose that $x_n \to x$ weakly in a normed space X. Show that $x \in \overline{\text{span}\{x_1, x_2, ...\}}$.

8.7. Suppose that $x_n \to x$ weakly in a normed space X. Show that there exists a sequence of linear combinations of the elements in the sequence (x_n) that converges to x in X.

8.8. Show that any closed subspace Y of a normed space X is *weakly closed*, that is, if (x_n) is a sequence in Y that converges weakly to x, then $x \in Y$.

8.9. Suppose that H is a Hilbert space. Show that $x_n \to x$ in X if and only if $x_n \to x$ weakly in H and $\|x_n\| \to \|x\|$.

8.10. Suppose that X and Y are two normed spaces and T is a bounded linear operator from X to Y. Show that if $x_n \to x$ weakly in X, then $Tx_n \to Tx$ weakly in Y.

8.11. A *weak Cauchy sequence* in a normed space X is a sequence (x_n) such that $(f(x_n))$ is a Cauchy sequence in $\mathbf{K}$ for every $f \in X'$. Show that every weak Cauchy sequence in X is bounded.

8.12. A normed space is said to be *weakly complete* if every weak Cauchy sequence is weakly convergent. Show that if X is reflexive, then X is weakly complete.

9. The Banach Fixed Point Theorem

9.1. Consider the function $f(x) = x/2 + x^{-1}$, $x \geq 1$. Show that f is a contraction and determine the contraction constant of f. Find the (unique) fixed point of f.

9.2. Consider the function $f(x) = x + x^{-1}$, $x \geq 1$. Show that f

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \geq 1 \text{ such that } x \neq y,$$

but that f has no fixed point.

9.3. (a) Write the following initial-value problem as an integral equation:

$$\begin{cases} x' = f(x, t), & t \geq 0 \\ x(0) = x_0 \end{cases}.$$

(b) Write the following initial-value problem as an integral equation:

$$\begin{cases} x'' = f(x, t), & t \geq 0 \\ x(0) = x_0 \\ x'(0) = x_1 \end{cases}.$$

9.4. Define $T : C([0,1]) \to C([0,1])$ by

$$Tx(t) = \int_0^t x(s)\, ds, \quad 0 \leq t \leq 1,$$

where $x \in C([0,1])$.

(a) Show that T is not a contraction on $C([0,1])$.

(b) Show that T^2 however is a contraction on $C([0,1])$.

(c) Deduce that T has a unique fixed-point and determine this fixed-point.

9.5. Define $T : L^2(0,1) \to L^2(0,1)$ by

$$Tx(t) = \int_0^t x(s)\,ds, \quad 0 \le t \le 1,$$

where $x \in C([0,1])$. Is T a contraction on $L^2(0,1)$?

9.6. Apply fixed-point iterations to the initial value problem

$$\begin{cases} x' = 1 + x^2, & t \ge 0 \\ x(0) = 0 \end{cases}$$

starting with $x_0 = 0$. Verify that the coefficients for $t, t^2, \dots, t^5$ in x_3 are the same as in the exact solution to the problem.

9.7. Consider the equation

$$x(t) - \mu \int_0^1 e^{t-s} x(s)\,ds = f(t), \quad 0 \le t \le 1,$$

where $\mu \in \mathbf{C}$ and $f \in C([0,1])$.

(a) For which $\mu \in \mathbf{C}$ does the equation have a unique solution $x \in C([0,1])$ for every right-hand side f?

(b) Solve the equation for as many values of μ as possible.

9.8. Suppose that $f \in C([0,1])$ and $\|f\|_\infty \le 1$. Show that the equation

$$x(t) + \mu \int_0^t t x^2(s)\,ds = f(t), \quad 0 \le t \le 1,$$

has a unique solution $x \in C([0,1])$ such that $\|x\|_\infty \le 2$ if $0 < \mu < \frac{1}{4}$.

10. Spectral Theory

10.1. Suppose that H is a separable Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. Define the linear operator T first on $(e_n)_{n=1}^\infty$ by

$$Te_n = e_{n+1}, \quad n = 1, 2, \dots,$$

then on H by linearity and continuity. Is T bounded? Show that T has no eigenvectors.

10.2. Find a bounded linear operator $T : C([0,1]) \to C([0,1])$ whose spectrum is a given interval $[a,b]$.

10.3. The operator $T : \ell^2 \to \ell^2$ is defined by $Tx = (\alpha_j x_j)_{j=1}^{\infty}$ for $x = (x_j)_{j=1}^{\infty} \in \ell^2$, where the sequence $(\alpha_j)_{j=1}^{\infty}$ is dense in $[0,1]$. Find $\sigma_p(T)$ and $\sigma(T)$.

10.4. Show that if T is a bounded operator on a normed space X, then $\|R_T(\lambda)\| \to 0$ as $\lambda \to \infty$.

10.5. Suppose that X is a complex Banach space, T is a bounded operator on X, and p is a polynomial. Show that the equation

$$p(T)x = y$$

has a unique solution $x \in X$ for every vector $y \in X$ if and only if $p(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$.

10.6. Suppose that X is a complex Banach space and T is a bounded operator on X. Show that

$$r_\sigma(\alpha T) = |\alpha| r_\sigma(T) \quad \text{and} \quad r_\sigma(T^k) = r_\sigma(T)^k$$

for every $\alpha \in \mathbf{C}$ and $k = 1, 2, \ldots$.

10.7. A bounded operator T on a complex Banach space X is said to be *idempotent* if $T^m = 0$ for some positive integer m. Find the spectrum of a idempotent operator T.

10.8. Suppose that X is a complex Banach space and T is a bounded operator on X. Deduce from the Spectral Radius formula that $r_\sigma(T) \leq \|T\|$.

10.9. Suppose that S and T are two bounded operators on a complex Banach space X that commute, i.e., $ST = TS$. Show that

$$r_\sigma(ST) \leq r_\sigma(S) r_\sigma(T).$$

10.10. Suppose that T is a normal operator on a Hilbert space H. Show that

$$r_\sigma(T) = \|T\|.$$

10.11. Suppose that A is a normed algebra with unit e. Show that every element of A, which has a left inverse and a right inverse, is in fact invertible. More precisely, show that if $x \in A$ and there exist $y, z \in A$ such that $yx = xz = e$, then x is invertible and $y = z = x^{-1}$.

10.12. Suppose that A is a normed algebra with unit e. Show that if $x \in A$ is invertible and commutes with $y \in A$, then also x^{-1} and y commute.

10.13. Suppose that A is a normed algebra. Show that if $\|x-e\| < 1$, then x is invertible and

$$x^{-1} = \sum_{n=0}^{\infty}(e-x)^n.$$

10.14. Suppose that A is a normed algebra. Show that if (x_n) and (y_n) are Cauchy sequences in A, then $(x_n y_n)$ is also a Cauchy sequence in A. Show furthermore that if $x_n \to x$ and $y_n \to y$ in A, then $x_n y_n \to xy$ in A.

10.15. The operator $T : \ell^2 \to \ell^2$ is defined by $Tx = (0, 0, x_1, x_2, ...)$ for $x = (x_j)_{j=1}^{\infty} \in \ell^2$. Is T bounded and, if so, what is the norm of T? Is T self-adjoint?

10.16. Suppose that S and T are two bounded operators on a complex Hilbert space H. Show that if S is self-adjoint, then the operator T^*ST is also self-adjoint.

10.17. Suppose that T is a linear operator on a Hilbert space H such that

$$(Tx, y) = (x, Ty) \quad \text{for all } x, y \in H.$$

Show that T is bounded.

10.18. Show that every compact, self-adjoint operator $T : H \to H$ on a complex Hilbert space H has at least one eigenvalue.

10.19. Show that every real, symmetric n by n matrix with positive elements has at least one positive eigenvalue.

Hints

1.11. Somewhere you will need to use Hölder's inequality

1.16. Consider the sequence $x_n = (\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{n}, 0, \dots),\ n = 1, 2, \dots$.

1.18. Since $x_n(t) \to 0$ for $0 \le t \le 1$, a good candidate for a limit is $x = 0$.

2.5. (a) Study the function $f(t) = 1 + t^\alpha - (1+t)^\alpha,\ 0 \le t < \infty$.
(b) Use (a).
(c) Use induction.

2.6. (a) Use Exercise 2.5.
(b) For the second part of the exercise, use the fact that

$$1 \le \frac{\|x\|_p}{\|x\|_\infty} = \left(\sum_{j=1}^{\infty} \left(\frac{|x_j|}{\|x\|_\infty} \right)^p \right)^{1/p}$$

together with the inequality $t^p \le t$, which holds for $0 \le t \le 1$.

3.10. Split the integral into two integrals.

3.11. Begin by changing variables.

3.14. $f(x) = \sum_{n=1}^{\infty} 2^{-n} \chi_{(r_n,1)}(x)$

4.2. Check the parallelogram law.

4.10. Apply the Cauchy–Schwarz inequality.

4.11. (a) Use the triangle inequality.

4.12. (b) Use the fact that $\|x\|_1 \ge \left| \int_0^1 x(t)\, dt \right| = 1$ for any $x \in M$.

5.7. (a) To find an example where the inequality is strict, consider projections in $\mathbf{R}^2$.

5.12. (a) To determine $\|T\|$, it can be useful to consider the sequence of functions $(x_n)_{n=1}^{\infty}$ defined by

$$x_n(t) = \frac{e^{-nt}}{n+1}, \quad 0 \le t \le 1.$$

5.13. Use a Hamel basis of X to define T.

7.3. Use the Riesz representation theorem for H.

7.4. Define first f on $\operatorname{span}\{x_0\} = \{tx_0 : t \in \mathbf{K}\}$. Then use the Hahn–Banach theorem to extend f to X.

7.7. Use Exercise 7.6

7.9. Apply the Uniform Boundedness Principle to the sequence (f_n) of linear functionals on ℓ^2, defined by $f_n(y) = \sum_{j=1}^{n} x_j y_j,\ y \in \ell^2$ for $n = 1, 2, \dots$.

7.11. Use the Uniform Boundedness Principle.

7.13. For the sufficiency part, use the Inverse Mapping theorem. For the necessity part, show that if $R(T) \ni y_n = Tx_n \to y \in Y$, then $y = Tx \in R(T)$.

7.14. (a) Show that if $x_n \in K$ for every n and $y_n = Tx_n \to y$, then $y = Tx$ for some $x \in K$. Use the compactness of K to extract a convergent subsequence to (x_n).

(b) Show that if $y_n = Tx_n \in K$ for every n and $x_n \to x$, then $y = Tx \in K$. Use the compactness of K to extract a convergent subsequence to (y_n).

7.15. It suffices according to Exercise 1.8 to show that $T^{-1}(F)$ is closed in X for every closed subset F of Y. Use Exercise 7.14 (b).

7.16. Use Exercise 7.14 (a).

8.1. Consider the functionals δ_{t_0} for $a \leq t_0 \leq b$.

8.3. (a) The dual of $\mathbf{c}_0$ is isomorphic to ℓ^1.

(b) If (x_n) were convergent in $\mathbf{c}_0$, then every sequence of coordinates would be convergent.

8.5. Use the Banach-Steinhaus theorem.

8.6. Suppose that $x \notin \overline{\text{span}\{x_1, x_2, ...\}}$ and use one of the corollaries to the Hahn–Banach theorem to produce a contradiction.

8.7. Use Exercise 8.6.

8.8. Use Exercise 8.7.

8.9. The necessity part is basically known. For the sufficiency part, expand $\|x - x_n\|^2$, using the definition of the norm.

8.10. This comes down to showing that $f(Tx_n) \to f(Tx)$ for every $f \in Y'$. To which space does $f \circ T$ belong?

8.11. Use the Banach–Steinhaus theorem.

9.5. The Cauchy–Schwarz inequality might come in handy.

10.4. Show that if $|\lambda| > \|T\|$, then

$$\|R_T(\lambda)\| \leq (|\lambda| - \|T\|)^{-1}.$$

10.7. Use the Spectral Radius formula.

10.9. Use the Spectral Radius formula.

10.17. One way of proving this is to use the Closed Graph theorem.

10.19. The trace of the matrix could be useful.

Answers

1.12. (b) No

1.18. (a) Yes
(b) No

1.19. (a) No
(b) Yes

2.1. (a) $\alpha = 1$
(b) $0 < \alpha \leq 1$

3.2. No

3.4. No

3.8. 0

3.9. 0

3.10. 3

3.11. $\ln 2$

3.13. $-\frac{1}{2} \ln 2$

3.14. $\sum_{n=1}^{\infty} 2^{-n}(1 - r_n)$

3.17. (b) No

4.2. No

4.4. (a) $Y^\perp = \{x \in \ell^2 : x_1 = x_3 = ... = 0\}$
(b) $Y^\perp = \{x \in \ell^2 : x_1 = ... = x_k = 0\}$

4.10. $x_j = c2^{-j/2},\ j = 1, 2, \ldots$, where $c \in \mathbf{C}$

4.13. (b) $x = 1$

4.14. (b) $\mathbf{c}_{00}^\perp = \{0\}$ and $\mathbf{c}_{00}^{\perp\perp} = \ell^2$.

4.15. (b) $M^\perp = \{\text{Konstanter}\}$ and $M^{\perp\perp} = M$.
(c) $y = \overline{x}$ and $z = x - \overline{x}$, where $\overline{x} = \int_0^1 x(t)\, dt$ is the average of x over $(0, 1)$.

5.6. 2

5.9. (a) $\|T\| = 1$
(c) $R(T) = \{y \in C^1([0, 1]) : y(0) = 0\}$. $R(T)$ is not closed in $C([0, 1])$.
(d) $T^{-1}y = y'$ for $y \in R(T)$. T^{-1} is unbounded.

5.10. (a) No

(b) Yes

(c) 0 and 1, respectively

5.11. (a) $\|T\| = 1$
(c) $R(T) = \{y \in \ell^\infty : (jy_j)_{j=1}^\infty \in \ell^\infty\}$. $R(T)$ is not closed in ℓ^∞.
(d) $T^{-1}y = (jy_j)_{j=1}^\infty$ for $y \in R(T)$. T^{-1} is unbounded.

5.12. (a) $\|T\| = 1$

(b) No

6.1. $\|f\| = 1$

7.1. $F = f$

7.2. Take $f(y) = y(0),\ y \in Y$. Then every $t_0 \in [0,1]$ gives a linear, norm-preserving extension F of f to $C([0,1])$, defined by $F(x) = x(t_0),\ x \in C([0,1])$.

7.3. If $f(y) = (y, a),\ y \in Y$, for some $a \in Y$, then $F(x) = (x, a),\ x \in H$, is the unique linear extension of f to H such that $\|F\| = \|f\|$.

7.12. (a) Yes

(b) No

8.3. (b) No

8.4. (b) No

9.1. The contraction constant is $\frac{1}{2}$ and the fixed point is $\sqrt{2}$.

9.3. (a) $x(t) = x_0 + \int_0^t f(x(s), s)\, ds,\ t \geq 0$

(b) $x(t) = x_0 + tx_1 + \int_0^t (t-s) f(x(s), s)\, ds,\ t \geq 0$

9.5. Yes

10.1. Yes

10.3. $\sigma_p(T) = \sigma(T) = [0,1]$

10.7. $\sigma(T) = \{0\}$

10.15. T is bounded and $\|T\| = 1$. T is not self-adjoint.