TATM85 – Applications of Functional analysis

Functional analysis = rigorous theory for solving problems in mathematical analysis and applications, where solutions are functions, not only numbers.

Function spaces, integrals and **operators** are fundamental and helpful in:

- Solving differential equations, simulations, image/signal processing, ...
- Representing and approximating functions (signals, images, flow, data, ...) efficiently in computers.
- Fourier series/transformations, wavelets, finite element method (FEM), ...
- In which sense do the approximations **converge** to their limiting functions, e.g. to solutions of differential equations and other problems ...?
  When are two functions "close to each other"?
- Theory of distributions and generalized functions (Dirac's delta-function, weak derivatives, Sobolev spaces ...)
- Various types of **convergence** in probability theory
  (almost surely, in probability, in distribution, weakly, ...)
- Observables in quantum mechanics = operators on Hilbert spaces.

Recall from the calculus:

- Continuity, derivatives and integrals are based on limits and convergence
- Use distances between points or numbers, e.g.

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

We shall consider distances much more generally, e.g. in spaces whose points are functions (function spaces).

Definition 0.1. $d : X \times X \to \mathbf{R}$ is a **metric** on a set X if $\forall x, y, z \in X$:

(i)	$d(x,y) \geq 0$	(nonnegative)
(ii)	$d(x,y) = 0$ iff $x = y$	(definite)
(iii)	$d(x,y) = d(y,x)$	(symmetric)
(iv)	$d(x,z) \leq d(x,y) + d(y,z)$	($\triangle$-inequality)

$X = (X, d)$ is called **metric space**

Definition 0.2. Ball with centre $x \in X$ and radius r is

$$B(x,r) := \{y \in X : d(x,y) < r\}$$

Also called r-neighbourhood of x.

Recall the following no(ta)tions:

- $[a,b] = \{x : a \leq x \leq b\}$ closed interval
- $(a,b) =]a,b[= \{x : a < x < b\}$ open interval
- (x,y) will also be used for points in $\mathbf{R}^2$ and later for the inner product between two vectors
- The supremum $\sup A$ of a set $A \subset \mathbf{R}$ is the smallest majorant of A, i.e. the smallest number $a \in [-\infty, \infty]$ such that $x \leq a$ for all $x \in A$.
- If $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers then

  $$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

  is the largest $a \in [-\infty, \infty]$ such that a subsequence of a_n converges to a.
- The infimum $\inf A$ and $\liminf\limits_{n\to\infty} a_n$ are defined similarly.
- The sequence $(a_n)_n$ converges if and only if

  $$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n =: \lim_{n\to\infty} a_n.$$

Some abbreviations:

pt, pts	point, points	spc, spcs	space, spaces
metr	metric	const	constant
acc	accumulation	isol	isolated
cont	continuous	fn/funct	function/functional
(tot) bdd	(totally) bounded	unbdd	unbounded
nbhd	neighbourhood	(sub)seq	(sub)sequence
conv	converge, convergent	div	divergent
unif	uniform	abs	absolutely
disj	disjoint	prod	product
separ	separable	compl	complete
ex	exist(s)	s.t.	such that
w.r.t	with respect to	TFAE	The following are equivalent
map	mapping	cpt	compact (set)
ineq	inequality	meas	measure, measurable
orthog	orthogonal	proj	projection
lin	linear	op	operator
int	integral	diff	differential
adj	adjoint	eigenv	eigenvector/eigenvalue
Lip	Lipschitz	Hilb	Hilbert

Thm	Theorem
Cor	Corollary (= consequence of a theorem)
Lemma	= simpler auxiliary theorem (hjälpsats)
Pf	Proof
Ex	Example

$\forall$	for all
$\exists$	there exists
$\in$	element of (belongs to)
$\emptyset$	empty set
$\subset$	subset (possibly equal)
$\subsetneq$	proper subset (not equal)
$\Longrightarrow$	implies
$\Longleftrightarrow$	equivalent to
$\nearrow$	increases to
$\searrow$	decreases to